IMS MATHS BOOK-05

VECTOR SPACES Set - I -

Field Let F be a non-empty set and $+^n$ and $\times^n$ be binary operations on F. Then algebraic structure $(F, +, \cdot)$ is said to be field if the following properties are satisfied.

(I) $(F, +)$ is an abelian group.

i) Closure property : $\forall a, b \in F \Rightarrow a+b \in F$

ii) Asso. prop :- $\forall a, b, c \in F \Rightarrow (a+b)+c = a+(b+c)$.

(iii) Existence of left identity : $\forall a \in F\ \exists 0 \in F$ s.t $0+a=a$

Here '0' is the identity elt.

(iv) Existence of left inverse :

$\forall a \in F, \exists -a \in F$ s.t $(-a)+a = 0$ (left identity)

Here $-a$ is the inverse of a in F.

(v) comm. prop : $\forall a, b \in F$; $a+b = b+a$

(II) $(F, \cdot)$ is an abelian group

i) Closure prop : $\forall a, b \in F \Rightarrow a \cdot b \in F$

ii) Asso. prop : $\forall a, b, c \in F \Rightarrow (a \cdot b) \cdot c = a \cdot (b \cdot c)$

iii) Existence of left identity :

$\forall a \in F\ \exists 1 \in F$ s.t $1 \cdot a = a$.

Here 1 is the identity in F.

(iv) Existence of left inverse :

$\forall 0 \neq a \in F\ \exists \frac{1}{a} \in F$ s.t $\frac{1}{a} \cdot a = 1$

$\therefore \frac{1}{a}$ is the inverse of a in F.

(v) Comm. prop : $\forall a, b \in F$; $ab = ba$

(III) $\times^n$ is distributive w.r.t $+^n$

i.e, $\forall a, b, c \in F \Rightarrow a \cdot (b+c) = ab + ac$.

Ex : $(I, +, \cdot)$ is not a field. Integers, not fractions ($a \cdot \frac{1}{a}$)? not integer

$(Q, +, \cdot)$, $(R, +, \cdot)$, $(C, +, \cdot)$ are fields.

$(Q^*, +, \cdot)$, $(R^*, +, \cdot)$, $(C^*, +, \cdot)$ are not fields.

$a+b=a$
Identity

Defn **Subfield:** Let F be a field and $K \subseteq F$.
If K is a field w.r.t same binary operations in F then K is called subfield of F.

Ex- I is not a subfield of Q

Integral field: I does not contain its fraction field

Q is a subfield of $\mathbb{R}$

$\mathbb{R}$ is " " $\mathbb{C}$

Defns **Internal Composition:**

Let A be any set. If $a * b \in A \ \forall a, b \in A$ then $*$ is said to be internal composition on A.

External Composition:

Let V and F be any two sets. If $a \circ \alpha \in V$ $\forall a \in F, \alpha \in V$ then '$\circ$' is said to be an external composition in V over F.

Vector Space or Linear Space

Let $(F, +, \cdot)$ be a field. The elts of F are called scalars. Let V be a non-empty set whose elts are called vectors. The following compositions are defined.

i) An internal composition in V called vector addition

(ii) An external composition in V over the field F called scalar multiplication.

If these compositions satisfy the following axioms then V is called vector space over the field F.

$\boxed{I}$. $(V, +)$ is an abelian group.

(i) Closure prop: $\forall \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

(ii) Asso. prop: $\forall \alpha, \beta, \gamma \in V \Rightarrow (\alpha + \beta) + \gamma = \{\alpha + (\beta + \gamma)\}$.

(iii) Existence of Identity:

$\forall \alpha \in V, \exists 0 \in V$ s.t $\alpha + 0 = 0 + \alpha = \alpha$

Here the identity elt $0 \in V$ is called zero vector.

(iv) Existence of inverse:

$\forall \alpha \in V, \exists -\alpha \in V$ s.t $\alpha + (-\alpha) = -\alpha + \alpha = 0$

(v) Comm. prop:

$\forall \alpha, \beta \in V \Rightarrow \alpha + \beta = \beta + \alpha$

$\boxed{\text{II}}$. The two compositions i.e, Scalar $\times^n$ and vector $+^n$.

$\forall\ a, b \in F;\ \alpha, \beta \in V \Rightarrow$

(i) $a \cdot (\alpha + \beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$

iv) $1\alpha = \alpha$; 1 is the unity elt of the field F.

Note: $\boxed{1}$. When V is a vector space over field F then we shall denote it by V(F) and we say ~~that~~ V(F) is a vector space.

$\boxed{2}$. If F is the field R of real no's then V is called real vector space. Similarly V(Q), V(C) are called rational, complex vector spaces respectively.

Problems: (1) $V = I$, $F = Q$

Is V(F) a vector space?

$I \subseteq Q$
$V \subseteq F$
$\therefore$ V is not a vec space

Sol^n Internal Composition:

$\forall \alpha, \beta \in I \Rightarrow \alpha + \beta \in I$

$\therefore$ vector $+^n$ is an internal composition on I.

External Composition:

$\forall a \in Q,\ \alpha \in I \Rightarrow a\alpha$ need not be an integer.

Ex $a = \frac{1}{2} \in Q,\ \alpha = 3 \in I \Rightarrow \frac{1}{2} \cdot 3 = \frac{3}{2} \notin I$.

$\therefore$ scalar $\times^n$ is not an external composition on I over Q.

$\therefore I(Q)$ is not a vector space

Note: If $V \subseteq F$ then $V(F)$ is not a vector space (except $V=\{0\} \subseteq F$)

(2) $V = \mathbb{R}$; $F = \mathbb{Q}$ $\quad \mathbb{Q} \subseteq \mathbb{R}$, $F \subseteq V$

Sol: $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha+\beta \in \mathbb{R}$.

and $\forall a \in \mathbb{Q} \subseteq \mathbb{R}$, $\alpha \in \mathbb{R} \Rightarrow a\alpha \in \mathbb{R}$

$\therefore$ Internal and external compositions are satisfied.

$\boxed{I}$ i) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha+\beta \in \mathbb{R}$

$\therefore$ Closure prop. is satisfied.

ii) $\forall \alpha, \beta, \gamma \in \mathbb{R}$

$\Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall \alpha \in \mathbb{R}$ $\exists$ $0 \in \mathbb{R}$ s.t $\alpha+0 = 0+\alpha = \alpha$

$\therefore$ Identity prop is satisfied..

$\therefore 0$ is identity elt.

(iv) $\forall \alpha \in \mathbb{R}$ $\exists$ $-\alpha \in \mathbb{R}$ s.t $\alpha+(-\alpha) = (-\alpha)+\alpha = 0$ (identity elt in $\mathbb{R}$)

$\therefore$ Inverse of α is $-\alpha$.

$\therefore$ Inverse prop. is satisfied.

(v) $\forall \alpha, \beta \in \mathbb{R} \Rightarrow \alpha+\beta = \beta+\alpha$

$\therefore$ Comm. prop. is satisfied

$\therefore (\mathbb{R}, +)$ is an abelian group.

$\boxed{II}$. $\forall a, b \in \mathbb{Q} \subseteq \mathbb{R}$; $\alpha, \beta \in \mathbb{R}$

(i) $a(\alpha+\beta) = a\alpha + a\beta$ (LDL in $\mathbb{R}$)

(ii) $(a+b)\alpha = a\alpha + b\alpha$ (RDL in $\mathbb{R}$)

(iii) $(ab)\alpha = a(b\alpha)$ (Asso. prop. in $\mathbb{R}$)

(iv) $1 \cdot \alpha = \alpha$ $\forall \alpha \in \mathbb{R}$. (1 is identity w.r.t x^n in $\mathbb{R}$)

$\therefore \mathbb{R}(\mathbb{Q})$ is vector space.

Note: If $F \subseteq V$ then $V(F)$ is a vector space.

Similarly $\mathbb{C}(\mathbb{Q})$, $\mathbb{C}(\mathbb{R})$ are also vector spaces

(3)

→ A field k can be regarded as a vector space over any subfield F of k.

Sol^n: Given that k is a field and F is a subfield of k.

$\therefore$ F is also field w.r.t some b-o^s defined in k.

Let us consider the elts of k as vectors.

$$\forall \alpha, \beta \in k \Rightarrow \alpha + \beta \in k.$$

and let us consider the elts of the subfield F as scalars.

Now $a \in F \subseteq k$, $\alpha \in k \Rightarrow a\alpha \in k$.

$\therefore$ Internal and external compositions are satisfied.

I. Since k is a field.

$\therefore$ $(k, +)$ is an abelian group

II. $\forall a, b \in F \subseteq k$; $\alpha, \beta \in k$

(i) $a(\alpha+\beta) = a\alpha + a\beta$ (LDL in k)

(ii) $(a+b)\alpha = a\alpha + b\alpha$ (RDL in k)

(iii) $(ab)\alpha = a(b\alpha)$ (Asso. prop. in k)

(iv) $1\alpha = \alpha$ — $\forall \alpha \in k$. and 1 is the identity elt of the subfield F.

($\because$ 1 is also identity elt of the field k).

$\therefore$ $1\alpha = \alpha$ $\forall \alpha \in k$.

$\therefore$ $K(F)$ is a vector space.

Note: If F is any field, then F itself is a vector space over the field F.

i.e, $F(F)$ is a vector space.

→ V = set of all vectors and F is a field of real no's.

Sol^n $\forall \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha} + \bar{\beta} \in V$ and

$a \in F, \bar{\alpha} \in V \Rightarrow a\bar{\alpha} \in V$

$\therefore$ Internal and external compositions are satisfied.

$\boxed{I}$ (i) $\forall\ \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha}+\bar{\beta} \in V$

$\therefore$ Closure prop. is satisfied

(ii) $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in V \Rightarrow (\bar{\alpha}+\bar{\beta})+\bar{\gamma} = \bar{\alpha}+(\bar{\beta}+\bar{\gamma})$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall\ \bar{\alpha} \in V\ \exists\ \bar{0} \in V$ s.t $\bar{\alpha}+\bar{0} = \bar{0}+\bar{\alpha} = \bar{\alpha}$

$\therefore \bar{0}$ is the identity vector in V.

iv) $\forall\ \bar{\alpha} \in V\ \exists -\bar{\alpha} \in V$ s.t $\bar{\alpha}+(-\bar{\alpha}) = (-\bar{\alpha})+\bar{\alpha} = \bar{0}$ (zero vector)

$\therefore$ inverse of $\bar{\alpha}$ is $-\bar{\alpha}$

(v) $\forall\ \bar{\alpha}, \bar{\beta} \in V \Rightarrow \bar{\alpha}+\bar{\beta} = \bar{\beta}+\bar{\alpha}$

$\therefore$ comm. prop. is satisfied.

$\boxed{II}$. $\forall\ a, b \in \mathbb{R}$ & $\bar{\alpha}, \bar{\beta} \in V$

(i) $a(\bar{\alpha}+\bar{\beta}) = a\bar{\alpha} + a\bar{\beta}$

(ii) $(a+b)\bar{\alpha} = a\bar{\alpha} + b\bar{\alpha}$

(iii) $(ab)\bar{\alpha} = a(b\bar{\alpha})$

(iv) $1\bar{\alpha} = \bar{\alpha}\ \forall\ \bar{\alpha} \in V$

$\therefore$ V(R) is a vector space.

$\rightarrow$ V = set of all $m \times n$ matrices with their elts as real numbers.

and $F = \mathbb{R}$.

Note: If V = the set of all $m\times n$ matrices with their elts as rational numbers and $F = \mathbb{R}$ then V(F) is not a vector space.

Because there is no external composition.

Ex: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix} \in V$; $\sqrt{7} \in \mathbb{R}$ then $\sqrt{7}A = \begin{bmatrix} \sqrt{7} & 2\sqrt{7} & 3\sqrt{7} \\ 0 & \sqrt{7} & 2\sqrt{7} \end{bmatrix} \notin V$.

($\because$ the elts of resulting matrix are not rational numbers)

→ Similarly, If V = the set of all $m \times n$ matrices with their elts as real numbers.

and $F = \mathbb{C}$ (complex numbers)

then $V(F)$ is not vector space

(4)

→ If V = the set of all $m \times n$ matrices with their elts as integers. and $F = \mathbb{Q}$ (rational numbers) then $V(F)$ is not a vector space.

→ V = the set of all ordered n-tuples and F is any field.

Soln. Let $V = \{(a_1, a_2, \ldots a_n) / a_1, a_2, \ldots a_n \in F\}$

Let $\alpha, \beta \in V$

Choose $\alpha = (a_1, a_2, \ldots a_n)$

$\beta = (b_1, b_2, \ldots b_n)$

where $a_1, a_2, \ldots a_n \in F$

$b_1, b_2 \ldots b_n \in F$

$\Rightarrow \alpha + \beta = (a_1, a_2 \ldots a_n) + (b_1, b_2, \ldots b_n)$

$= (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n) \in V$

$\because (a_1 + b_1, a_2 + b_2 \ldots a_n + b_2 \in F)$

∴ Internal composition is satisfied.

and $a \in F$, $\alpha \in V$

$\Rightarrow a\alpha = a(a_1, a_2 \ldots a_n)$

$= (aa_1, aa_2 \ldots aa_n) \in V$

$\because aa_1, aa_2 \ldots aa_n \in F$

∴ External composition is satisfied.

$\boxed{I}$. (i) $\forall \alpha, \beta \in V$

$\Rightarrow \alpha + \beta = (a_1 + b_1, a_2 + b_2, \ldots a_n + b_n) \in V$

$\because a_1 + b_1, a_2 + b_2, \ldots a_n + b_n \in V$

(ii) $\forall \alpha, \beta, \gamma \in V$

$(\alpha + \beta) + \gamma = [(a_1, a_2 \ldots a_n) + (b_1, b_2, \ldots b_n)] + (c_1, c_2 \ldots c_n)$

$= (a_1+b_1, a_2+b_2, \dots a_n+b_n) + (c_1, c_2 \dots c_n)$

$= ((a_1+b_1)+c_1, (a_2+b_2)+c_2, \dots (a_n+b_n)+c_n)$

$= (a_1+(b_1+c_1), a_2+(b_2+c_2), \dots, a_n+(b_n+c_n))$

(by asso. prop. w.r.t + in F)

$= (a_1, a_2 \dots a_n) + (b_1+c_1, b_2+c_2, \dots b_n+c_n)$

$= (a_1, a_2, \dots a_n) + [(b_1, b_2 \dots b_n) + (c_1, c_2 \dots c_n)]$

$= \alpha + (\beta + \gamma)$

$\therefore$ Asso. prop. is satisfied.

(iii) We have $0 = (0, 0, \dots 0) \in V$ where $0 \in F$

if $\alpha = (a_1, a_2, \dots a_n) \in V$ where $a_1, a_2, \dots a_n \in F$

then $0 + \alpha = (0, 0, 0, \dots 0) + (a_1, a_2, \dots a_n)$

$= (0+a_1, 0+a_2, \dots 0+a_n)$

$= (a_1, a_2 \dots a_n)$

$= \alpha$

Similarly $\alpha + 0 = \alpha$.

$\therefore 0 + \alpha = 0 + \alpha = \alpha$

$\therefore 0 = (0, 0, 0 \dots 0)$ is the identity elt in V.

(iv) If $\alpha = (a_1, a_2, \dots a_n) \in V$ where $a_1, a_2 \dots a_n \in F$

then $-\alpha = -(a_1, a_2, \dots a_n)$

$= (-a_1, -a_2, \dots -a_n) \in V$

where $-a_1, -a_2, \dots -a_n \in F$

Now $(-\alpha) + \alpha = ((-a_1)+a_1, (-a_2)+a_2, \dots (-a_n)+a_n)$

$= (0, 0, \dots 0)$

Similarly $\alpha + (-\alpha) = 0$

$\therefore (-\alpha) + \alpha = \alpha + (-\alpha) = 0$ $\therefore -\alpha$ is the inverse of α in V.

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(v) $\forall \alpha, \beta \in V \Rightarrow \alpha+\beta = (a_1, a_2, \dots a_n) + (b_1, b_2, \dots b_n)$

$= (a_1+b_1, a_2+b_2, \dots\dots a_n+b_n)$

$= (b_1+a_1, b_2+a_2, \dots\dots b_n+a_n)$

$= (b_1\ b_2 \dots b_n) + (a_1\ a_2 \dots a_n)$

$= \beta+\alpha.$

$\therefore (V,+)$ is an abelian group.

$\boxed{II}$ for $\alpha, \beta \in V$; $a, b \in F$

(i) $a(\alpha+\beta) = a[(a_1, a_2, \dots\dots a_n) + (b_1, b_2, \dots b_n)]$

$= a[a_1+b_1, a_2+b_2, \dots\dots a_n+b_n]$

$= (a(a_1+b_1), a(a_2+b_2) \dots a(a_n+b_n))$

$= (aa_1+ab_1, aa_2+ab_2, \dots aa_n+ab_n)$ (By LDL in F)

$= (aa_1, aa_2 \dots aa_n) + (ab_1, ab_2, \dots ab_n)$

$= a(a_1, a_2, \dots a_n) + a(b_1, b_2, \dots b_n)$

$= a\alpha + a\beta.$

(ii) $(a+b)\alpha = (a+b)(a_1, a_2, \dots a_n)$

$= ((a+b)a_1, (a+b)a_2, \dots (a+b)a_n)$

$= (aa_1+ba_1, aa_2+ba_2, \dots aa_n+ba_n)$

$= (aa_1, aa_2 \dots aa_n) + (ba_1, ba_2 \dots ba_n)$ (By RDL in F)

$= a(a_1\ a_2 \dots a_n) + b(a_1\ a_2 \dots a_n)$

$= a\alpha + b\alpha$

(iii) $(ab)\alpha = (ab)(a_1\ a_2 \dots a_n)$

$= ((ab)a_1, (ab)a_2, \dots (ab)a_n)$

$= (a(ba_1), a(ba_2), \dots a(ba_n))$ (By asso. prop in F)

$= a(ba_1, ba_2 \dots ba_n)$

$= a[b(a_1, a_2, \dots a_n)]$

$= a(b\alpha).$

(iv) $1\alpha = 1(a_1, a_2, \ldots a_n)$

$= (1a_1, 1a_2 \ldots 1a_n)$

$= (a_1, a_2, \ldots a_n)$ $\left(\because 1 \in F,\ a_i \in F \Rightarrow 1a_i \in F \right.$

$= \alpha \quad \forall \alpha \in V.$ $\left. \therefore 1a_1, 1a_2 \ldots 1a_n \in F. \right)$

$\therefore$ V(F) is a vector space.

Note: $\boxed{1}$ → The vector space of all ordered n-tuples over F is denoted by $V_n(F)$.

→ Sometimes denote it by $F^{(n)}$ or F^n.

$\therefore V_n(F)$ or $F^{(n)} = \{(a_1, a_2, \ldots a_n) / a_1, a_2 \ldots a_n \in F\}$

$\boxed{2}$. $V_2(F) = \{(a_1, a_2) / a_1, a_2 \in F\}$ is a vector space of all ordered pairs over F.

Similarly $V_3(F) = \{(a_1, a_2, a_3) / a_1, a_2, a_3 \in F\}$ is the vector space of all ordered triples or triads over F.

→ $F[x]$ = The set of all polynomials and ~~F is any~~ field.

Sol: $F[x] = \{ f(x) / f(x) = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \ldots \ldots ;\ a_0, a_1, \ldots \ldots a_n \ldots \in F \}$

Now $f(x), g(x) \in F[x]$

$\therefore$ Choose $f(x) = \sum_{i=0}^{\infty} a_i x^i$

$g(x) = \sum_{i=0}^{\infty} b_i x^i$; where $a_i, b_i \in F$, $i = 0, 1, 2, \ldots$

$\Rightarrow f(x) + g(x) = (a_0 x^0 + a_1 x + a_2 x^2 + \ldots) + (b_0 + b_1 x + b_2 x^2 + \ldots)$

$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \ldots$

$= \sum (a_i + b_i) x^i \in F[x]$. $\because a_i + b_i \in F,\ i = 0, 1, 2, \ldots$

$\therefore$ External Composition is satisfied.

Now $f(x) \in F[x]$; $a \in F$

$$af(x) = a(a_0 + a_1 x + a_2 x^2 + \cdots)$$
$$= aa_0 + (aa_1)x + (aa_2)x^2 + \cdots$$
$$= \Sigma (aa_i) x^i \in F[x] \qquad \left(\because a, a_i \in F,\ i = 0,1,2,\ldots \Rightarrow aa_i \in F\right)$$

$\therefore$ External composition is satisfied.

$\boxed{I}$ (i) $\forall f(x), g(x) \in F[x]$, where $f(x) = \Sigma a_i x^i$, $g(x) = \Sigma b_i x^i$; $a_i, b_i \in F$, $i = 0,1,2,\ldots$

$$\Rightarrow f(x) + g(x) = \Sigma a_i x^i + \Sigma b_i x^i$$
$$= \Sigma (a_i + b_i) x^i \in F[x] \qquad [\because a_i + b_i \in F]$$

(ii) $\forall f(x), g(x), h(x) \in F[x]$

$$\Rightarrow [f(x) + g(x)] + h(x) = \left[\Sigma a_i x^i + \Sigma b_i x^i\right] + \Sigma c_i x^i$$
$$= \Sigma (a_i + b_i) x^i + \Sigma c_i x^i$$
$$= \Sigma [(a_i + b_i) + c_i] x^i$$
$$= \Sigma [a_i + (b_i + c_i)] x^i \quad (\because \text{By asso. prop in } F)$$
$$= \Sigma a_i x^i + \Sigma (b_i + c_i) x^i$$
$$= \Sigma a_i x^i + \left[\Sigma b_i x^i + \Sigma c_i x^i\right]$$
$$= f(x) + [g(x) + h(x)]$$

$\therefore$ Asso. prop. is satisfied.

(iv) We have $\underset{\text{(zero polynomial)}}{0} = 0 + 0x + 0x^2 + \cdots$

$$= \Sigma 0x^i \in F[x] \quad ;\ 0 \in F$$

if $f(x) = \Sigma a_i x^i \in F[x]$; $a_i \in F$, $i = 0,1,2,\ldots$

then $0 + f(x) = \Sigma 0x^i + \Sigma a_i x^i$

$$= \Sigma (0 + a_i) x^i$$
$$= \Sigma a_i x^i$$
$$= f(x)$$

Similarly $f(x)+0 = f(x)$.

$\therefore 0+f(x) = f(x)+0 = f(x) \quad \forall f(x) \in F[x]$.

$\therefore$ Identity elt is the zero polynomial

(iv) If $f(x) \in F[x]$ then $-f(x) \in F[x]$.

i.e. $f(x) = a_0 + a_1 x + \cdots \in F[x]$; $a_0, a_1, a_2 \cdots \in F$

then $-f(x) = -a_0 + (-a_1)x + (-a_2)x^2 + \cdots \in F[x]$;

$-a_0, -a_1, -a_2 \cdots \in F$.

we have

$$(-f(x)) + f(x) = (-a_0+a_0) + (-a_1+a_1)x + (-a_2+a_2)x^2 + \cdots$$
$$= 0 + 0x + 0x^2 + \cdots$$
$$= 0 \text{ (zero polynomial)}$$

Similarly $f(x) + (-f(x)) = 0$

$\therefore (-f(x)) + f(x) = f(x) + (-f(x)) = 0$

$\therefore -f(x)$ is the inverse polynomial of $f(x)$ in $F[x]$

(v) If $f(x), g(x) \in F[x]$

$$\Rightarrow f(x)+g(x) = (a_0+b_0)x^0 + (a_1+b_1)x + \cdots$$
$$= (b_0+a_0)x^0 + (b_1+a_1)x + \cdots$$
(By asso. prop in F)
$$= (b_0 + b_1 x + \cdots) + (a_0 + a_1 x + \cdots)$$
$$= g(x) + f(x)$$

$\therefore$ Commutative property is satisfied.

$\therefore (F[x], +)$ is an abelian group.

$\boxed{\text{II}}$: $\forall f(x), g(x) \in F[x]$; $a, b \in F$

(i) $a(f(x)+g(x)) = a\left[(a_0 + a_1 x + a_2 x^2 + \cdots) + (b_0 + b_1 x + b_2 x^2 + \cdots)\right]$

$$= a\left[(a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2 + \cdots\right]$$
$$= a(a_0+b_0) + a(a_1+b_1)x + a(a_2+b_2)x^2 + \cdots$$
$$= (aa_0 + ab_0) + (aa_1 + ab_1)x + (aa_2 + bb_2)x^2 + \cdots$$
(By asso. prop. in F)
$$= \left[(aa_0) + (aa_1)x + (aa_2)x^2 + \cdots\right] + \left[(ab_0) + (ab_1)x + (ab_2)x^2 + \cdots\right]$$

$= a(a_0 + a_1 x + a_2 x^2 + \cdots) + a(b_0 + b_1 x + b_2 x^2 + \cdots)$

$= a\, f(x) + a\, g(x)$

(ii) $(a+b) f(x) = (a+b)(a_0 + a_1 x + a_2 x^2 + \cdots)$

$= [(a+b) a_0] + [(a+b) a_1] x + [(a+b) a_2] x^2 + \cdots$

$= (a a_0 + b a_0) + (a a_1 + b a_1) x + \cdots$

$= (a a_0 + (a a_1) x + \cdots) + (b a_0 + (b a_1) x + \cdots)$

$= a(a_0 + a_1 x + \cdots) + b(a_0 + a_1 x + \cdots)$

$= a\, f(x) + b\, f(x)$.

(iii) Similarly

(iii) $(ab) f(x) = a\, b f(x)$

(iv) $1\, f(x) = f(x)$. ~~$\forall f(x) \in F[x]$~~.

→ Let F be the field and let P_n be the set of all polynomials (of degree atmost n) over the field F. S.T P_n is vector space over the field F.

Solⁿ Let $P_n = \{ f(x) / f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ where $a_0, a_1, \ldots a_n \in F \}$

$\forall f(x), g(x) \in P_n$;

Chose $f(x) = a_0 + a_1 x + \cdots + a_n x^n$

$g(x) = b_0 + b_1 x + \cdots + b_n x^n$

$a_0, a_1, \ldots a_n ; b_1 b_2 \ldots b_n \in F$.

$\Rightarrow f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1) x + \cdots + (a_n + b_n) x^n$

$\in P_n$ $(\because a_0 + b_0, a_1 + b_1, \ldots a_n + b_n \in F$ and polynomial of degree atmost n)

$\forall f(x) \in P_n$; $c \in F$

$\Rightarrow c f(x) = c a_0 + (c a_1) x + (c a_2) x^2 + \cdots + (c a_n) x^n$.

$\in P_n$ $(\because c a_0, c a_1, \ldots c a_n \in F$ and polynomial of degree atmost n)

$\therefore$ Internal and external composition are satisfied.

[I] (i) $\forall\, f(x), g(x) \in P_n \Rightarrow f(x)+g(x) \in P_n$

$\therefore$ Closure prop. is satisfied.

(ii) $\forall\, f(x), g(x), h(x) \in P_n$

$$\Rightarrow \big(f(x)+g(x)\big)+h(x) = f(x)+\big(g(x)+h(x)\big)$$

$\therefore$ Asso. prop. is satisfied.

(iii) $\forall\, f(x) \in P_n \;\exists\; I(x) = 0+0x+0x^2+\cdots+0x^n \in P_n$ over F.

s.t $f(x)+I(x) = f(x)$.

Similarly $I(x)+f(x) = f(x)$

$\therefore f(x)+I(x) = I(x)+f(x) = f(x) \quad \forall\, f(x) \in P_n$

$\therefore I(x)$ is the identity polynomial in P_n.

iv) Inverse prop.

(v) Commutative prop:

$\therefore (P_n, +)$ is an abelian group.

[II]. $\forall\, f(x), g(x) \in P_n$; $a, b \in F$

we have (i) $a(f(x)+g(x)) = a\big[(a_0+a_1x+a_2x^2+\cdots a_nx^n) + (b_0+b_1x+\cdots b_nx^n)\big]$

$$= a\big[(a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n\big]$$

$$= a(a_0+b_0)+a(a_1+b_1)x+\cdots+a(a_n+b_n)x^n$$

$$= (aa_0+ab_0)+(aa_1+ab_1)x+\cdots(aa_n+ab_n)x^n \quad \text{(By LDL in } F)$$

$$= (aa_0+aa_1x+\cdots+aa_nx^n)+(ab_0+(ab_1)x+\cdots(ab_n)x^n)$$

$$= a(a_0+a_1x+\cdots+a_nx^n)+a(b_0+b_1x+\cdots+b_nx^n)$$

$$= af(x)+a\,g(x).$$

(ii) Sly $(a+b)f(x) = af(x)+bf(x)$

(iii) Sly $(ab)f(x) = a(bf(x))$

(v) Sly $1f(x) = f(x) \quad \forall\, f(x) \in P_n$. $\therefore P_n(F)$ is a vector space

→ Let F be any field and S be any non-empty set. Let V be the set of all functions from S to F. (8)

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i.e. $V = \{ f \mid f : S \to F \}$

Let us define sum of two vectors f and g in V as follows.

$(f+g)(x) = f(x) + g(x) \quad \forall x \in S$

and the product of the scalar 'c' in F and the function f in V as follows:

$(cf)(x) = c\,f(x) \quad \forall x \in S$.

then V(F) is vector space.

Sol^n

$\forall f, g \in V \Rightarrow (f+g)x = f(x) + g(x) \quad \forall x \in S$ (By defn)

Since $f(x), g(x) \in F$ and F is a field.

$\Rightarrow f(x) + g(x) \in F$.

$(f+g)(x) = f(x) + g(x) \in F$

$\therefore (f+g) : S \to F$

$\therefore f+g \in V$.

Internal Composition is satisfied.

$\forall f \in V, c \in F \Rightarrow (cf)(x) = c\,f(x) \quad \forall x \in S$ (By defn)

Since $f(x) \in F$, $c \in F$ and F is a field.

$\therefore c\,f(x) \in F$

$\therefore cf : S \to F$.

$\Rightarrow cf \in V; \ \forall c \in F, f \in V$

External composition is satisfied.

$\boxed{I}$ (i) $\forall f, g \in V \Rightarrow f+g \in V$. Closure prop. is satisfied

(ii) $\forall f, g, h \in V$

$\Rightarrow [(f+g)+h](x) = (f+g)(x) + h(x) = (f(x)+g(x)) + h(x)$
$= f(x) + [g(x) + h(x)]$

$= [f+(g+h)](x)$ By asso. prop in F i.e, $f(x), g(x), h(x) \in F$ $\Rightarrow [f(x)+g(x)]+h(x) = f(x)+[g(x)+h(x)]$

$\therefore (f+g)+h = f+(g+h)$

(iii) If $\forall x \in S$, $I(x) = 0 \in F$ then $I \in V$. (i.e $I: S \to F$)

now $(I+f)(x) = I(x)+f(x)$
$= 0+f(x)$
$= f(x) \quad \forall f(x) \in F$

$\therefore I+f = f \quad \forall f \in V$

Sly $f+I = f \quad \forall f \in V$

$\therefore I+f = f+I = f \quad \forall f \in V$

$\therefore$ Identity elt $= I \in V$.

(iv) if $f \in V$, then $-f = (-1)f \in V$.

Now $[f+(-f)](x) = f(x)+(-f)(x)$
$= f(x)+[-1 f(x)]$
$= f(x) - f(x)$
$= 0 = I(x)$.

$\therefore f+(-f) = 0 = I$

Sly $(-f)+f = 0 = I$

$\therefore f+(-f) = -f+f = 0 = I$

$\therefore$ Inverse of f is $-f$ in V.

(v) $\forall f, g \in V \Rightarrow (f+g)(x) = f(x)+g(x)$ (By defn)
$= g(x)+f(x)$ By asso. in F
$= (g+f)(x)$ i.e, $f(x), g(x) \in F$ $\Rightarrow f(x)+g(x) = g(x)+f(x)$

$\therefore f+g = g+f$

$\therefore$ Commutative prop is satisfied.

$\boxed{II}$ $\forall a, b \in F$, $f, g \in V$

(i) $[a(f+g)](x) = a(f+g)(x)$ (By defn)
$= a(f(x)+g(x))$ (By defn)
$= af(x)+ag(x)$ (By LDL in F)
$= (af)(x)+(ag)(x) = (af+ag)(x)$ $\therefore a(f+g) = af+ag$

(ii) $[(a+b)f](x) = (a+b)f(x)$ (By defn)
$= af(x) + bf(x)$
$= (af)(x) + (bf)(x)$
$= (af+bf)(x)$
$\therefore (a+b)f = af+bf.$

(iii) $[(ab)f](x) = (ab)f(x)$ (by defn)
$= a(bf(x))$
$= a(bf)(x)$

$\therefore (ab)f = a(bf)$

(iv) $(1 \cdot f)(x) = 1 \cdot f(x)$ (By defn)
$= f(x)$ By identity in F
$\forall f(x) \in F$
$\therefore 1f = f \quad \forall f \in V$
$\therefore V(F)$ is a vector space

→ Let V be the set of all pairs (x, y) of real numbers and let F be the field of real numbers.
Examine in each of the following cases whether V is a vector space over the field of real numbers or not ?

(1) $(x, y) + (x_1, y_1) = (x+x_1, y+y_1)$
$c(x, y) = (cx, y)$

not II(iv) fail (2) $(x, y) + (x_1, y_1) = (x+x_1, 0)$
$c(x, y) = (cx, 0)$

not II(ii) fail (3) $(x, y) + (x_1, y_1) = (x+x_1, y+y_1)$
$c(x, y) = (|c|x, |c|y)$

not II(i) fail (4) $(x, y) + (x_1, y_1) = (x+x_1, y+y_1)$
$c(x, y) = (c^2x, c^2y)$

qst (5) $(x,y)+(x_1,y_1) = (x+x_1, y+y_1)$
II (M) fail
$c(x,y) = (0, cy)$.

Sol^n ⇒ (I) $\boxed{I}$ Let $\alpha = (x,y)$; $\beta = (x_1, y_1) \in V$
where $x, y, x_1, y_1 \in \mathbb{R}$

(i) $\alpha+\beta = (x,y)+(x_1,y_1)$
$= (x+x_1, y+y_1) \in V$ $(\because x+x_1, y+y_1 \in \mathbb{R})$
∴ closure prop. is satisfied

(ii) $(\alpha+\beta)+\gamma = [(x,y)+(x_1,y_1)]+(x_2,y_2)$
$= (x+x_1, y+y_1)+(x_2,y_2)$
$= ((x+x_1)+x_2, (y+y_1)+y_2)$ (By defn)
$= (x+(x_1+x_2), y+(y_1+y_2))$; (By asso. prop in R)
$= (x,y)+(x_1+x_2, y_1+y_2)$
$= (x,y)+[(x_1,y_1)+(x_2,y_2)]$
$= \alpha+(\beta+\gamma)$
∴ Asso. prop. is satisfied

(iii) $\forall \alpha = (x,y) \in V$ $\exists \bar{0} = (0,0) \in V$, $0 \in \mathbb{R}$
s.t $\alpha+0 = (x,y)+(0,0)$
$= (x+0, y+0)$
$= (x,y)$
$= \alpha$
Sly $0+\alpha = \alpha$.
∴ $0+\alpha = \alpha+0 = \alpha$
∴ $(0,0)$ is the identity in V.

(iv) $\forall \alpha \in V$ $\exists -\alpha = (-x,-y) \in V$; $-x, -y \in \mathbb{R}$.
$= (x,y)$
s.t $\alpha+(-\alpha) = (x,y)+(-x,-y)$

$= (x-x, y-y)$ (By defn)

$= (0,0)$

Hly $(-\alpha)+\alpha = (0,0)$

$\therefore \alpha+(-\alpha) = (-\alpha)+\alpha = (0,0)$

$\therefore -\alpha$ is the inverse of α.

(v) $\forall\ \alpha, \beta \in V \Rightarrow \alpha+\beta = \beta+\alpha$ (by defn)

$\therefore$ comm. prop. is satisfied.

$\therefore (V,+)$ is an abelian group.

$\boxed{\text{II}}$ (a) $\forall\ \alpha, \beta \in V$; $a, b \in \mathbb{R}$

(i) $a(\alpha+\beta) = a[(x,y)+(x_1,y_1)]$

$= a(x+x_1, y+y_1)$ (By defn)

$= (a(x+x_1), y+y_1)$ (by defn) —(1)

and $a\alpha + a\beta = a(x,y) + a(x_1,y_1)$

$= (ax, y) + (ax_1, y_1)$

$= (a(x+x_1), y+y_1)$ —(2)

$\therefore$ from (1) & (2)

$a(\alpha+\beta) = a\alpha + a\beta$.

(ii) $(a+b)\alpha = (a+b)(x,y)$

$= ((a+b)x, y)$ —(1)

and $a\alpha + b\alpha = a(x,y) + b(x,y)$

$= (ax, y) + (bx, y)$

$= ((a+b)x, 2y)$ —(2)

$\therefore$ from (1) & (2) we have

$(a+b)\alpha \neq a\alpha + b\alpha$.

$\therefore V(\mathbb{R})$ is not a vector space.

Sol^n (2) II (ii)

Let $\alpha = (x,y) \in V$; $x,y \in \mathbb{R}$

then $1\alpha = 1(x,y) = (1x, 0)$ (By defn)

$= (x, 0)$

But $(x, 0) \neq (x,y)$ (if $y \neq 0$)

$\therefore 1\alpha \neq \alpha \ \forall \alpha \in V$

$\therefore V(\mathbb{R})$ is not a vector space.

→ Let $V(F)$ be a vector space and $\vec{0}$ be the zero vector of V. Then

(i) $a\vec{0} = \vec{0}$ $\forall a \in F$

(ii) $0\alpha = \vec{0}$ $\forall \alpha \in V$

(iii) $a(-\alpha) = -(a\alpha)$ $\forall a \in F, \forall \alpha \in V$.

(iv) $(-a)\alpha = -(a\alpha)$ $\forall a \in F, \forall \alpha \in V$

(v) $a(\alpha - \beta) = a\alpha - a\beta$ $\forall a \in F$, and $\forall \alpha, \beta \in V$.

(vi) $a\alpha = 0 \Rightarrow a = 0$ (or) $\alpha = 0$

proof:

(i) we have $a0 = a(0+0)$ $\quad (\because 0 = 0+0)$

$= a0 + a0$ $\quad (\because a(\alpha+\beta) = a\alpha + a\beta$, $a \in F, \alpha, \beta \in V)$

$\Rightarrow a0 + 0 = a0 + a0$

$(\because a0 \in V$ and $0 + a0 = a0)$

Given that $V(F)$ be a vector space

$\therefore$ V is an abelian group w.r.t addition

Therefore by right cancellation law in V,

we get $0 = a0$

$\Rightarrow \boxed{a0 = 0}$ $\forall a \in F$.

(ii). we have $0\alpha = (0+0)\alpha$ $\quad (\because 0 = 0+0)$

$= 0\alpha + 0\alpha$

$\Rightarrow 0 + 0\alpha = 0\alpha + 0\alpha$

$(\because 0\alpha \in V$ and $0 + 0\alpha = 0\alpha)$

Since V is an abelian group w.r.t addition

Therefore by right cancellation law in V,

we get $0 = 0\alpha$

$\therefore 0\alpha = 0 \quad \forall \alpha \in V$.

(iii) we have $a[\alpha + (-\alpha)] = a\alpha + a(-\alpha)$

$\Rightarrow a0 = a\alpha + a(-\alpha)$

$\Rightarrow 0 = a\alpha + a(-\alpha)$

$\Rightarrow a(-\alpha)$ is the additive inverse of $a\alpha$

$\Rightarrow a(-\alpha) = -a\alpha$

$\therefore a(-\alpha) = -a\alpha \quad \forall a \in F, \ \forall \alpha \in V$

(iv) we have $[a + (-a)]\alpha = a\alpha + (-a)\alpha$

$\Rightarrow 0\alpha = a\alpha + (-a)\alpha$

$\Rightarrow 0 = a\alpha + (-a)\alpha$

$\Rightarrow (-a)\alpha$ is the additive inverse of $a\alpha$.

$\Rightarrow (-a)\alpha = -a\alpha$

$\therefore (-a)\alpha = -a\alpha \quad \forall a \in F, \ \forall \alpha \in V$.

(v) we have $a(\alpha - \beta) = a(\alpha + (-\beta))$

$= a\alpha + a(-\beta)$

$= a\alpha + [-(a\beta)] \quad (\because a(-\beta) = -a\beta)$

$= a\alpha - a\beta$

$\therefore a(\alpha - \beta) = a\alpha - a\beta \quad \forall a \in F, \ \forall \alpha, \beta \in V$.

(vi) Let $a\alpha = 0$ and $a \neq 0$.

Then a^{-1} exists because a is a non-zero element of the field F.

$\therefore a\alpha = 0 \Rightarrow a^{-1}(a\alpha) = a^{-1}0$

$\Rightarrow (a^{-1}a)\alpha = 0$

$\Rightarrow 1\alpha = 0$

$\Rightarrow \alpha = 0$

Again let $a\alpha = 0$ and $\alpha \neq 0$.

Then to prove that $a = 0$.

If possible suppose that $a \neq 0$.

Then a^{-1} exists

$\therefore a\alpha = 0 \Rightarrow a^{-1}(a\alpha) = a^{-1}0$

$\Rightarrow (a^{-1}a)\alpha = 0$

$\Rightarrow 1\alpha = 0$

$\Rightarrow \alpha = 0$

Then we get a contradiction

~~that~~ α must be a zero vector.

Therefore a must be equal to 0.

Hence $\alpha \neq 0$ and $a\alpha = 0$

$\Rightarrow a = 0$

$\therefore a\alpha = 0 \Rightarrow a = 0$ or $\alpha = 0$

→ Let $V(F)$ be a vector space. Then

(i) If $a, b \in F$ and α is a non-zero vector of V, we have $a\alpha = b\alpha \Rightarrow a = b$

(ii) If $\alpha, \beta \in V$ and a is a non-zero element of F, we have

$a\alpha = a\beta \Rightarrow \alpha = \beta$.

Proof: (i) we have $a\alpha = b\alpha$

$$\Rightarrow a\alpha - b\alpha = 0$$
$$\Rightarrow (a-b)\alpha = 0$$
$$\Rightarrow a-b = 0$$
$$\Rightarrow a = b$$

(ii) we have $a\alpha = a\beta$

$$\Rightarrow a\alpha - a\beta = 0$$
$$\Rightarrow a(\alpha-\beta) = 0$$
$$\Rightarrow \alpha-\beta = 0 \text{ ; Since } a \neq 0$$
$$\Rightarrow \alpha = \beta$$

→ On R^n, define two operations

$$\alpha \oplus \beta = \alpha - \beta$$
$$c\cdot\alpha = -c\alpha$$

The operations on the right are the usual ones. which of the axioms for a vector space are satisfied by $(R^n, \oplus, \cdot)$?

→ Let V be the set of all complex-valued functions f on the real line such that (for all t in R)

$$f(-t) = \overline{f(t)}.$$

The bar denotes complex conjugation. Show that V, with the operations

$$(f+g)(t) = f(t) + g(t)$$
$$(cf)(t) = c f(t)$$

is a vector space over the field of real numbers. Give an example of a function in V which is not real-valued.

→ Let R^+ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows:

$u+v = u \cdot v$ for all $u, v \in R^+$

$\alpha u = u^\alpha$ for all $u \in R^+$ and real scalar α.

prove that R^+ is a real vector space.

→ which of the following subsets of V_4 are vector spaces for coordinatewise addition and scalar multiplication ?

The set of all vectors $(x_1, x_2, x_3, x_4) \in V_4$ such that

(a) $x_4 = 0$ (b) $x_1 = 0$ (c) $x_2 > 0$ (d) $x_3^2 \geq 0$ (e) $x_1^2 < 0$

(f) $2x_1 + 3x_2 = 0$ (g) $x_1 + \frac{2}{3}x_2 - 3x_3 + x_4 = 1$. (h) $x_1 = 1$

Ans: (a), (b), (f) and (d) are vector spaces.

→ which of the following subsets of P are vector spaces ?

The set of all polynomials p such that

(a) degree of $p \leq n$ (b) degree of $p = 3$

(c) degree of $p \geq 4$ (d) $p(1) = 0$

(e) $p(2) = 1$ (f) $p'(1) = 0$

(g) p has integral coefficients.

Ans: (a), (d) and (f) are vector spaces.

Notations:

$\mathcal{C}[a,b]$ = the set of all real-valued functions defined and continuous on the closed interval $[a,b]$.

$\mathcal{C}^{(1)}[a,b]$ = the set of all real-valued functions defined on $[a,b]$ and whose first derivatives are continuous on $[a,b]$.

$\mathcal{C}^{(n)}[a,b]$ = the set of all real-valued functions defined on $[a,b]$, differentiable n-times and whose n^{th} derivatives are continuous on $[a,b]$. These functions are called n-times continuously differentiable functions.

→ Which of the following subsets $\mathcal{C}[0,1]$ are vector spaces?

The set of all functions $f \in \mathcal{C}[a,b]$ such that

yes (a) $f(1/2) = 0$ (b) $f(3/4) = 0$ yes (c) $f'(x) = x f(x)$

yes (d) $f(0) = f(1)$ (e) $f(x) = 0$ at a finite number of points in $[0,1]$

yes (f) f has a local minima at $x = 1/2$

(g) f has a local extrema at $x = 1/2$.

Ans: (a), (c), (d) & (f) are vector spaces.

10(i)

→ Is Z_7 a vector space over Z_5 ?

Sol^n: NO.

$Z_5 = \{0,1,2,3,4\}$ is not subfield of

$Z_7 = \{0,1,2,3,4,5,6\}$

because:

$2+3=0$ in Z_5 but $2+3 \neq 0$ in Z_7.

Hence Z_5 is not a vector space over Z_5.

→ Let $K = Z_3$, the integers modulo 3. How many elements are there in the vector space $V = K^4$?

Sol^n: There are three choices 0, 1 or 2, for each of the four components of a vector in V.

Hence V has $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ elements.

→ Can $\mathbb{C}^2$ (pairs of complex numbers) be defined as a vector space: (a) over $\mathbb{R}$? (b) over $\mathbb{Q}$? (c) over $\mathbb{C}$? (d) over $\mathbb{Z}$?

Sol^n: (a), (b), (c) are vector space.

where as (d) is not a vector space.

because $\mathbb{Z}$ is not a field.

→ Can $\mathbb{R}^n$ be defined as a vector space: (a) over $\mathbb{Q}$ (b) over $\mathbb{R}$ (c) over $\mathbb{C}$?

Sol^n: (a), (b) are vector spaces.

where as (c) is not a vector space

because $\mathbb{C}$ is not a subfield of $\mathbb{R}$.

→ Let $V = \{ \langle a_n \rangle : a_n \in \mathbb{R} \ \forall n \in \mathbb{N} \}$ i.e., V is the set of all real sequences. Prove that V is a vector space over $\mathbb{R}$, where addition and scalar-multiplication are defined component wise.

Miscellaneous results and notations

→ Let $\mathcal{F}(I)$ be the set of all real-valued functions defined on the interval I.
with pointwise addition and scalar multiplication $\mathcal{F}(I)$ becomes a real vector space.

The zero of this space is the function 0 given by $0(x) = 0$ for all $x \in I$.

→ Note: If, instead of the real valued function we use complex valued functions defined on I and pointwise addition and scalar multiplication, then we get a complex vector space (using complex scalars).

we denote this complex vector space by $\mathcal{F}_c(I)$.

→ Let $\mathcal{P}(I)$ denote the set of all polynomials p with real coefficients defined on the interval I.

where p is a function whose value at x is $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n$ for all $x \in I$;

where α_i's are real numbers and n is a non-negative integer.

Using pointwise addition and scalar multiplication as for functions, we find that $\mathcal{P}(I)$ is a real vector space.

If we take complex coefficients for the

polynomials and use complex scalars, then we get the complex vector space $P_c(I)$.

In both cases the vector space 0 of the space is the zero polynomial given by

$$0(x) = 0 \text{ for all } x \in I$$

→ $\mathcal{C}[a,b]$, $\mathcal{C}^{(1)}[a,b]$, $\mathcal{C}^{(n)}[a,b]$ are real vector spaces under pointwise addition and scalar multiplication.

∴ we have sum of two continuous (differentiable) functions is continuous (differentiable) and any scalar multiple of a continuous (differentiable) function is continuous (differentiable).

By changing the domain of definitions of continuity and differentiability to the open interval (a,b), we get, similarly the real vector space $\mathcal{C}(a,b)$ and $\mathcal{C}^{(n)}(a,b)$ for each positive integer n.

Note: By changing real-valued functions to complex-valued functions and using complex scalars, we get the complex vector spaces $\mathcal{C}_c[a,b]$ and $\mathcal{C}_c(a,b)$

→ Let $\mathcal{C}^{\infty}[a,b]$ stand for the set of all functions defined on $[a,b]$ and having derivatives of all orders on $[a,b]$. This is a real vector space for the usual operations. It is called the space of infinitely differentiable functions on $[a,b]$.

SUBSPACE:

Let $V(F)$ be a vector space and $W \subseteq V$ if W is a vector space w.r.t the internal and external compositions in V then W is called a subspace of V.

Theorem:

→ $V(F)$ is a vector space, W is a subset of V ($W \subseteq V$); W is a subspace of $V(F)$ iff the internal and external compositions are satisfied in W.

i.e. (i) $\forall \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$

(ii) $\forall a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

proof: Necessary part:

Let W be a subspace of $V(F)$.

∴ By defn W is a vector space w.r.t the internal and external compositions in V.

∴ Internal and external compositions are satisfied in W.

i.e., (i) $\forall \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$ and

(ii) $\forall \alpha \in W, a \in F \Rightarrow a\alpha \in W$.

Sufficient condition:

Let $W \subseteq V$ and internal and external compositions be satisfied in W.

i.e, (i) $\forall \alpha, \beta \in W \Rightarrow \alpha + \beta \in W$

(ii) $\forall \alpha \in W, a \in F \Rightarrow a\alpha \in W$.

~~Hence~~.

$\boxed{I}$. (i) $\forall \alpha, \beta \in W \subseteq V \Rightarrow \alpha + \beta \in W$. (by hypothesis)

∴ Closure prop. is satisfied.

(ii) $\forall \alpha, \beta, \gamma \in W \subseteq V \Rightarrow (\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ by Asso. prop. in V.

∴ Asso. prop. is satisfied

(iii) $\forall \alpha, \beta \in W \subseteq V$

$\Rightarrow \alpha+\beta = \beta+\alpha$. (By comm. prop in V)

$\therefore$ comm. prop. is satisfied in W.

(iv) Take $a = 0 \in F$, $\alpha \in W \Rightarrow a\alpha = 0\alpha \in W$ (by hyp)

$\Rightarrow 0 \in W$

$\therefore 0+\alpha = \alpha+0 = \alpha \quad \forall \alpha \in W \subseteq V$ (By identity prop in V.)

$\therefore$ Identity satisfied in W.

(v) $1 \in F \Rightarrow -1 \in F$.

Take $a = -1 \in F$; $\alpha \in V$

$\Rightarrow a\alpha = (-1)\alpha \in W$ (By hyp)

$\Rightarrow -\alpha \in W$

$\therefore \alpha+(-\alpha) = (-\alpha)+\alpha = 0$ (By inverse prop. in V)

$\therefore$ inverse of α is $-\alpha$.

$\therefore (W, +)$ is an abelian group.

$\boxed{\text{II}}$ $\forall \alpha, \beta \in W \subseteq V$, $a, b \in F$

(i) $a(\alpha+\beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$

iv) $1\alpha = \alpha \quad \forall \alpha \in W \subseteq V$

By axioms w.r.t external compositions in V

$\therefore W(F)$ is a vector space.

$\therefore W(F)$ is a subspace of $V(F)$.

$\rightarrow$ $V(F)$ is a vector space, W is a subset of $V(F)$ (i.e., $W \subseteq V$); W is a subspace of $V(F)$ iff $a, b \in F$ and $\alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$.

proof: N.C:

Let W be a subspace of $V(F)$.

$\therefore$ By defn W is a vector space w.r.t internal and external compositions in V.

$\therefore a, b \in F$, $\alpha, \beta \in W$ ~~$\Rightarrow a\alpha, b\beta \in W$~~

$\Rightarrow a\alpha, b\beta \in W$ (By external composition in W)

$\Rightarrow a\alpha + b\beta \in W$ (by internal comp in W)

$\boxed{I}$ **S.C**

(i) Take $a = b = 1 \in F$

$1 \in F,\ \alpha, \beta \in W \subseteq V \Rightarrow 1\alpha + 1\beta \in W$ (By hyp)

$\Rightarrow \alpha + \beta \in W$

Closure prop. is satisfied.

(ii) $\forall \alpha, \beta, \gamma \in W \subseteq V$

$\Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$ (By asso. prop. in V)

(iii) $\forall \alpha, \beta \in W \subseteq V$

$\Rightarrow \alpha + \beta = \beta + \alpha$ (By comm. prop. in V)

$\therefore$ Asso. prop and Comm. prop are satisfied in W.

(iv) Take $a = b = 0 \in F$

$0 \in F,\ \alpha, \beta \in W \subseteq V$

$\Rightarrow 0\alpha + 0\beta \in W$ (by hyp)

$\Rightarrow 0 \in W$

$\therefore \forall \alpha \in W \subseteq V \ \exists\ 0 \in W$ s.t

$\alpha + 0 = \alpha = 0 + \alpha$ (By identity in V)

$\therefore$ 0 is identity elt in W.

(v) $1 \in F \Rightarrow -1 \in F$

Take $a = -1 \in F$ & $b = 0 \in F$

$\alpha, \beta \in W \subseteq V$

$\Rightarrow (-1)\alpha + 0\beta \in W$ (by hyp)

$\Rightarrow -\alpha \in W$

$\therefore$ If $\alpha \in W \subseteq V$ then $-\alpha \in W \subseteq V$

$\therefore \alpha + (-\alpha) = \alpha + (-\alpha) = 0$ (By inverse axiom in V)

$\therefore$ inverse of α is $-\alpha$

$\therefore (W, +)$ is an abelian group.

$\boxed{II}$. $\forall \alpha, \beta \in W \subseteq V;\ a, b \in F$

(i) $a(\alpha+\beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$; (iv) $1\alpha = \alpha \ \forall \alpha \in W \subseteq V$

By axioms w.r.t external compositions in V.

$\therefore$ $W(F)$ is a vector space

$\therefore$ $W(F)$ is a subspace of $V(F)$.

$\rightarrow$ $V(F)$ is a vectorspace ; $W \subseteq V$; W is a subspace of $V(F)$ iff (i) $\forall\ \alpha, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

Proof: N.C :- Let W be a subspace of V.

$\therefore$ By defn W is a vector space w.r.t the internal and external compositions in V.

By internal composition :

$\forall \alpha, \beta \in W \Rightarrow \alpha \in W, -\beta \in W$ (By inverse axiom in W)

$\Rightarrow \alpha + (-\beta) \in W$ (By closure prop in W)

$\Rightarrow \alpha - \beta \in W$

By external composition

$a \in F, \alpha \in W \Rightarrow a\alpha \in W$.

S.C :- Let $W \subseteq V$; (i) $\forall \alpha, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) $\forall\ a \in F, \alpha \in W \Rightarrow a\alpha \in W$

$\boxed{\text{I}}$. (i) Take $a = 0 \in F$

$0 \in F, \alpha \in W \subseteq V$

$\Rightarrow 0\alpha \in W$ (by hyp)

$\Rightarrow 0 \in W$.

$\therefore$ $0 + \alpha = \alpha + 0 = \alpha \quad \forall \alpha \in W \subseteq V$ (by Identity axiom of V)

$\therefore$ Identity prop. is satisfied in W.

and '0' is the identity in W.

(ii) Take $\alpha = 0 \in W$, $\beta = \alpha \in W$

$\Rightarrow 0 - \alpha \in W$ (by hyp)

$\Rightarrow -\alpha \in W$

$\therefore$ $\alpha + (-\alpha) = (-\alpha) + \alpha = 0$ (by inverse axiom of V)

$\therefore$ Inverse prop. is satisfied in W. and inverse of α is $-\alpha$

(iii) $\alpha, \beta \in W \subseteq V \Rightarrow \alpha, -\beta \in W$ — $(\because \alpha \in W \Rightarrow -\alpha \in W)$

$\Rightarrow \alpha - (-\beta) \in W$

$\Rightarrow \alpha + \beta \in W$

$\therefore$ closure prop. is satisfied in W.

(iv) $\forall \alpha, \beta, \gamma \in W \subseteq V \Rightarrow \alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

$\therefore$ Asso. prop. is satisfied.

(v) $\forall \alpha, \beta \in W \subseteq V \Rightarrow \alpha + \beta = \beta + \alpha$

$\therefore$ comm. prop is satisfied.

$\therefore$ $(W,+)$ is an abelian group.

$\boxed{II}$. $\forall \alpha, \beta \in W \subseteq V$, $a, b \in F$

(i) $a(\alpha + \beta) = a\alpha + a\beta$
(ii) $(a+b)\alpha = a\alpha + b\alpha$
(iii) $(ab)\alpha = a(b\alpha)$

} By axioms w.r.t external composition in V

(iv) $1\alpha = \alpha \quad \forall \alpha \in W \subseteq V$

$\therefore$ $W(F)$ is a vector space.

$\therefore$ $W(F)$ is a subspace of $V(F)$

→ $V(F)$ is a vector space and $W \subseteq V$; W is a subspace of $V(F)$ iff $a \in F$, $\alpha, \beta \in W \Rightarrow a\alpha + \beta \in W$.

$(\Longleftrightarrow)$

Proof

N.C :-

Let W be a subspace of $V(F)$.

$\therefore$ By defn W is a vector space w.r.t the internal & external compositions in V.

By external composition in W —

$\forall a \in F, \alpha \in W \Rightarrow a\alpha \in W$

By internal composition

$\forall a\alpha \in W, \beta \in W \Rightarrow a\alpha + \beta \in W$.

S.C :- Let $W \subseteq V$ &

$\forall \alpha, \beta \in W, a \in F \Rightarrow a\alpha + \beta \in W$

$\boxed{I}$ (i) Take $a = 1 \in F$

$1 \in F,\ \alpha, \beta \in W \Rightarrow 1 \cdot \alpha + \beta \in W$ (by hyp)

$\Rightarrow \alpha + \beta \in W$

$\therefore$ Closure prop. is satisfied in W

(ii) $\forall \alpha, \beta, \gamma \in W \subseteq V \Rightarrow (\alpha+\beta)+\gamma = \alpha+(\beta+\gamma)$

(iii) $\alpha+\beta = \beta+\alpha$

$\therefore$ Asso. & comm. prop. is satisfied in W.

(iv) $1 \in F, \Rightarrow -1 \in F$

Take $a = -1 \in F$, $\beta = \alpha \in W$

$-1 \in F,\ \alpha, \alpha \in W \Rightarrow (-1)\alpha + \alpha \in W$ (by hyp)

$\Rightarrow -\alpha + \alpha = 0 \in W$

$\therefore 0 + \alpha \Rightarrow \alpha + 0 = \alpha \quad \forall \alpha \in W \subseteq V$

(by identity of V)

$\therefore$ Identity prop is satisfied in W.

0 is the identity in W.

(v) $-1 \in F;\ \alpha, 0 \in W \Rightarrow -1 \cdot \alpha + 0 \in W$ (by hyp)

$\Rightarrow -\alpha \in W$

$\therefore \alpha + (-\alpha) = (-\alpha) + \alpha = 0$ (By inverse axiom of V)

$\therefore$ Inverse prop is satisfied in W

$\therefore -\alpha$ is inverse of α.

$\therefore (W, +)$ is an abelian group.

$\boxed{II}$ $\forall \alpha, \beta \in W \subseteq V,\ a, b \in F$

(i) $a(\alpha+\beta) = a\alpha + a\beta$

(ii) $(a+b)\alpha = a\alpha + b\alpha$

(iii) $(ab)\alpha = a(b\alpha)$

(iv) $1\alpha = \alpha \quad \forall \alpha \in W \subseteq V$

$\therefore W(F)$ is a vector space

$\therefore W(F)$ is a subspace of $V(F)$

Algebra of subspaces

→ The intersection of any two subspaces of a vector space $V(F)$ is also a subspace of $V(F)$.

proof: Let W_1 & W_2 be any two subspaces of $V(F)$.

Let $W = W_1 \cap W_2$

$a, b \in F$; $\alpha, \beta \in W$

$\Rightarrow a, b \in F$; $\alpha, \beta \in W_1 \cap W_2$

$\Rightarrow a, b \in F$; ($\alpha, \beta \in W_1$ and $\alpha, \beta \in W_2$)

$\Rightarrow a\alpha + b\beta \in W_1$ and $a\alpha + b\beta \in W_2$

($\because W_1$ & W_2 are two subspaces)

$\therefore W_1 \cap W_2$ is also subspace of $V(F)$

$\therefore$ The intersection of two subspaces is also a subspace.

→ The arbitrary intersection of subspaces i.e, the intersection of any family of subspaces of a vector space is also a subspace.

proof Let $W_1, W_2, \dots$ be the given family of subspaces of the vector space $V(F)$.

Let $W = W_1 \cap W_2 \cap \dots\dots$

$= \bigcap_{i \in N} W_i \quad (i = 1, 2, \dots)$

$a, b \in F$, $\alpha, \beta \in W \Rightarrow a, b \in F$; $\alpha, \beta \in \bigcap_{i \in N} W_i$

$\Rightarrow a, b \in F$, $\alpha, \beta \in W_i \quad \forall i \in N$

$\Rightarrow a\alpha + b\beta \in W_i \quad \forall i \in N$ ($\because W_i$ is a subspace for all $i \in N$)

$\Rightarrow a\alpha + b\beta \in \bigcap_{i \in N} W_i = W$

$\therefore W = \bigcap_{i \in N} W_i$ is a subspace of $V(F)$

$\therefore$ The intersection of any family of subspaces of a vector space is also a subspace.

→ The union of two subspaces of a vector space need not be a subspace

Sol[n] $V_3(F) = \{(a_1, a_2, a_3) / a_1, a_2, a_3 \in F\}$ is a vector space

Let $W_1 = \{(0, a, b) / a, b \in F\} \subseteq V_3$

and $W_2 = \{(x, 0, y) / x, y \in F\} \subseteq V_3$

$a_1, a_2 \in F$;

$\alpha = (0, a, b)$,

$\beta = (0, c, d) \in W_1$

$a, b, c, d \in F$

$\Rightarrow a_1\alpha + a_2\beta = a_1(0, a, b) + a_2(0, c, d)$

$= (0, a_1a, a_1b) + (0, a_2c, a_2d)$

$= (0, a_1a + a_2c, a_1b + a_2d) \in W_1$

$\therefore W_1$ is a subspace. $(\because 0, a_1a + a_2c, a_1b + a_2d \in W_1)$

Now $a_1, a_2 \in F$; $\alpha = (x_1, 0, y_1)$, $\beta = (x_2, 0, y_2) \in W_2$

$x_1, y_1, x_2, y_2 \in F$

$\Rightarrow a_1\alpha + a_2\beta = a_1(x_1, 0, y_1) + a_2(x_2, 0, y_2)$

$= (a_1x_1, 0, a_1y_1) + (a_2x_2, 0, a_2y_2)$

$= (a_1x_1 + a_2x_2, 0, a_1y_1 + a_2y_2) \in W_2$

$\therefore W_2$ is a subspace of $V(F)$ $(\because a_1x_1 + a_2x_2, 0, a_1y_1 + a_2y_2 \in W_2)$

If $F = Q$ then we have

$(0, \frac{1}{2}, 3) \in W_1$, $(1, 0, 3) \in W_2$

$\Rightarrow (0, \frac{1}{2}, 3), (1, 0, 3) \in W_1 \cup W_2$

$\Rightarrow (0, \frac{1}{2}, 3) + (1, 0, 3) = (1, \frac{1}{2}, 6) \notin W_1 \cup W_2$

$(\because$ neither $(1, \frac{1}{2}, 6) \in W_1$ nor $(1, \frac{1}{2}, 6) \in W_2)$

$W_1 \cup W_2 = \{(0, \frac{1}{2}, 3), (1, 0, 3)\}$

$\therefore W_1 \cup W_2$ is not closed under vector addition.

$\therefore W_1 \cup W_2$ is not a subspace of $V_3(F)$.

→ The union of two subspaces is a subspace iff one is contained in the other.

proof: Let W_1 and W_2 be two subspaces of the vector space $V(F)$.

N.C. Let $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

$W_1 \subseteq W_2 \Rightarrow W_1 \cup W_2 = W_2$ (subspace of $V(F)$)

$W_2 \subseteq W_1 \Rightarrow W_1 \cup W_2 = W_1$ (subspace of $V(F)$)

$\therefore W_1 \cup W_2$ is a subspace of $V(F)$.

S.C: Let $W_1 \cup W_2$ be a subspace of $V(F)$

then we prove that $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

If possible suppose that $W_1 \not\subseteq W_2$ or $W_2 \not\subseteq W_1$

if $W_1 \not\subseteq W_2$
let $\alpha \in W_1$ then $\alpha \notin W_2$

if $W_2 \not\subseteq W_1$
let $\beta \in W_2$ then $\beta \notin W_1$

Now $\alpha \in W_1$, $\beta \in W_2$
$\Rightarrow \alpha, \beta \in W_1 \cup W_2$
$\Rightarrow \alpha + \beta \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is a subspace)
$\Rightarrow \alpha + \beta \in W_1$ or $\alpha + \beta \in W_2$

Now $\alpha + \beta \in W_1$, $\alpha \in W_1$
$\Rightarrow (\alpha + \beta) - \alpha \in W_1$ ($\because W_1$ is a subspace)
$\Rightarrow \beta \in W_1$
which is contradiction to $\beta \notin W_1$

and $\alpha + \beta \in W_2$, $\beta \in W_2$
$\Rightarrow (\alpha + \beta) - \beta \in W_2$ ($\because W_2$ is a subspace)
$\Rightarrow \alpha \in W_2$
which contradiction to $\alpha \notin W_2$

$\therefore$ Our assumption that $W_1 \not\subseteq W_2$ or $W_2 \not\subseteq W_1$ is wrong

$\therefore W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

<u>Note</u>: [1] Let $V(F)$ be any vector space.
Then V itself and the subset of V consisting of the zero vector alone are always subspaces V.
These two subspaces are called improper subspaces.
If V has any other subspace then it is called a proper subspace.

[2]. The subspace of V consisting of the zero vector only is called the zero subspace of V.

Problems :

$\Rightarrow$ Let $W = \{ (a_1, a_2, 0) / a_1, a_2 \in F \} \subseteq V_3(F)$.
Then S.T W is a subspace of $V_3(F)$.

Sol: Let $a, b \in F$; $\alpha, \beta \in W$

Choose $\alpha = (a_1, a_2, 0)$
$\beta = (b_1, b_2, 0)$
where $a_1, a_2, b_1, b_2 \in F$

$$\Rightarrow a\alpha + b\beta = a(a_1, a_2, 0) + b(b_1, b_2, 0)$$
$$= (aa_1, aa_2, 0) + (bb_1, bb_2, 0)$$
$$= (aa_1 + bb_1, aa_2 + bb_2, 0) \in W$$

$\therefore W$ is a subspace of $V_3(F)$. $\quad aa_1 + bb_1, aa_2 + bb_2 \in F$

$\rightarrow$ Let $W = \{ (x_1, x_2, x_3) / a_1x_1 + a_2x_2 + a_3x_3 = 0$
a_1, a_2, a_3 are fixed elts in F
$x_1, x_2, x_3 \in F \} \subseteq V_3(F)$.

S.T W is a subspace of $V_3(F)$.

Sol: $\forall a, b \in F$; $\alpha, \beta \in W$

Choose $\alpha = (x_1, x_2, x_3)$; $a_1x_1 + a_2x_2 + a_3x_3 = 0$
$\beta = (y_1, y_2, y_3)$; $a_1y_1 + a_2y_2 + a_3y_3 = 0$

$$\Rightarrow a\alpha + b\beta = a(x_1, x_2, x_3) + b(y_1, y_2, y_3)$$
$$= (ax_1, ax_2, ax_3) + (by_1, by_2, by_3)$$
$$= (ax_1 + by_1, ax_2 + by_2, ax_3 + by_3)$$

and $a_1(ax_1 + by_1) + a_2(ax_2 + by_2) + a_3(ax_3 + by_3)$
$$= a(a_1x_1 + a_2x_2 + a_3x_3) + b(a_1y_1 + a_2y_2 + a_3y_3)$$
$$= a(0) + b(0)$$
$$= 0$$

$\therefore a\alpha + b\beta \in W$

$\therefore W$ is a subspace of $V_3(F)$.

→ P.T the set of all solutions (a, b, c) of the equation $a+b+2c=0$ is a subspace of the vector space $V_3(\mathbb{R})$.

Sol^n Let $W = \{(a, b, c) / a+b+2c=0;\ a, b, c \in \mathbb{R}\} \subseteq V_3(\mathbb{R})$.

Let $a, b \in \mathbb{R}$, $\alpha, \beta \in W$

Choose $\alpha = (a_1, b_1, c_1)$; $a_1+b_1+2c_1=0$

$\beta = (a_2, b_2, c_2)$; $a_2+b_2+2c_2=0$

where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{R}$

→ S.T the set W of the elts of the vector space $V_3(\mathbb{R})$ of the form $(x+2y, y, -x+3y)$ where $x, y \in \mathbb{R}$ is a subspace of $V_3(\mathbb{R})$.

Sol^n Let $W = \{(x+2y, y, -x+3y) / x, y, z \in \mathbb{R}\} \subseteq V_3(\mathbb{R})$

let $a, b \in F$; $\alpha, \beta \in W$

Choose $\alpha = (x_1+2y_1, y_1, -x_1+3y_1)$

$\beta = (x_2+2y_2, y_2, -x_2+3y_2)$

$\Rightarrow a\alpha+b\beta = a(x_1+2y_1, y_1, -x_1+3y_1) + b(x_2+2y_2, y_2, -x_2+3y_2)$

$= (ax_1+2ay_1, ay_1, -ax_1+3ay_1) + (bx_2+2by_2, by_2, -bx_2+3by_2)$

$= (ax_1+bx_2+2(ay_1+by_2), ay_1+by_2, -[ax_1+bx_2]+3(ay_1+by_2))$

$\in W$.

$\therefore a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $V_3(\mathbb{R})$.

→ which of the following sets of vectors $\alpha = (a_1, a_2, \ldots a_n)$ in $\mathbb{R}^n$ are subspace of $\mathbb{R}^n$?

(i) all α s.t $a_1 \le 0$

(ii) all α s.t a_3 is an integer

(iii) all α s.t $a_2 + 4a_3 = 0$

(iv) all α s.t $a_1 + a_2 + \cdots + a_n = k$ (constant)

Solⁿ (i) Let $W = \{\alpha / \alpha = (a_1, a_2, \ldots a_n) \text{ and } a_1 \le 0\} \subseteq \mathbb{R}^n$ i.e. $V_n(\mathbb{R})$

If $a_1 = -3$ then $a_1 < 0$

Let $\alpha = (-3, a_2, a_3, \ldots a_n) \in W$

and if $a = -2 \in \mathbb{R}$

then $a\alpha = -2(-3, a_2 \ldots a_n)$

$= (6, -2a_2, \ldots -2a_n) \notin W$ $\left(\because \underset{a_1=}{6} > 0.\right)$

$\therefore \forall \alpha \in W, a \in \mathbb{R} \Rightarrow a\alpha \notin W$

$\therefore W$ is not a subspace of $\mathbb{R}^n$.

(ii) Let $W = \{\alpha / \alpha = (a_1, a_2, \ldots a_n) \text{ and } a_3 \text{ is an integer}\} \subseteq \mathbb{R}^n$

If $a_3 = -3$ is an integer.

Let $\alpha = (a_1, a_2, -3, \ldots a_n) \in W$

and $a = \frac{1}{2} \in \mathbb{R}$

then $a\alpha = \left(\frac{a_1}{2}, \frac{a_2}{2}, -\frac{3}{2}, \frac{a_4}{2}, \ldots a_n\right) \notin W$

$(\because -3/2$ is not an integer$)$.

$\therefore \forall \alpha \in W, a \in \mathbb{R} \Rightarrow a\alpha \notin W$.

$\therefore W$ is not a subspace of $\mathbb{R}^n$

(iii) Let $W = \{\alpha / \alpha = (a_1, a_2, \ldots a_n) \text{ and } a_2 + 4a_3 = 0\} \subseteq \mathbb{R}^n$

Now $a, b \in \mathbb{R}$, $\alpha, \beta \in W$

Choose $\alpha = (a_1, a_2, \ldots a_n)$ and $a_2 + 4a_3 = 0$

$\beta = (b_1, b_2, \ldots b_n)$ and $b_2 + 4b_3 = 0$

$\Rightarrow a\alpha+b\beta = a(a_1,a_2,a_3,\ldots a_n) + b(b_1,b_2,b_3,\ldots b_n)$

$= (aa_1+bb_1, aa_2+bb_2, aa_3+bb_3, \ldots aa_n+bb_n)$ — (1)

Now we have

$(aa_2+bb_2)+4(aa_3+bb_3) = a(a_2+4a_3)+b(b_2+4b_3)$

$= a(0)+b(0)$

$= 0$

$\therefore$ (1) $\equiv a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $\mathbb{R}^n$.

(iv) Let $W = \{\alpha / \alpha = (a_1,a_2,\ldots a_n)$ and $a_1+a_2+\cdots+a_n = k$ (const)$\} \subseteq \mathbb{R}^n$

Let $a,b \in F$, $\alpha,\beta \in W$.

Choose $\alpha = (a_1,a_2,\ldots a_n)$ and $a_1+a_2+\cdots+a_n = k$

$\beta = (b_1,b_2,\ldots b_n)$ and $b_1+b_2+\cdots+b_n = k$.

$\Rightarrow a\alpha+b\beta = a(a_1,a_2,\ldots a_n)+b(b_1,b_2,\ldots b_n)$

$= (aa_1+bb_1, aa_2+bb_2, \ldots aa_n+bb_n)$ — (1)

Now we have

$(aa_1+bb_1)+(aa_2+bb_2)+\cdots+(aa_n+bb_n)$

$= a(a_1+a_2+\cdots+a_n)+b(b_1+b_2+\cdots+b_n)$

$= ak+bk$

$= (a+b)k.$

If $k=0$ then (1) $\equiv a\alpha+b\beta \in W$

$\therefore W$ is a subspace of $\mathbb{R}^n$

If $k \neq 0$ then $a\alpha+b\beta \notin W$.

$\therefore W$ is not a subspace of $\mathbb{R}^n$.

$\rightarrow$ S.T W is not a subspace of $\mathbb{R}^3 = V$, where $W = \{(a,b,c) / a^2+b^2+c^2 \leq 1\} \subseteq V$.

Solⁿ Let $\alpha = (0,1,0)$, $\beta = (1,0,0) \in W$

then $\alpha+\beta = (1,1,0) \notin W$ $(\because 1^2+1^2+0 = 2 > 1)$

$\therefore W$ is not a subspace of $V = \mathbb{R}^3$.

→ S.T W is not subspace of $V = \mathbb{R}^3$.
where $W = \{(a, b, c) / a, b, c \in \mathbb{Q}\} \subset \mathbb{R}^3$.

Sol^n^ Let $a_1 = \sqrt{2} \in \mathbb{R}$, $\alpha = (1, 2, 3) \in W$

$$\Rightarrow a_1\alpha = \sqrt{2}(1, 2, 3)$$
$$= (\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}) \notin W$$

$\therefore W$ is not a subspace of V. $(\because \sqrt{2}, 2\sqrt{2}, 3\sqrt{2} \notin \mathbb{Q})$

→ S.T W is not a subspace of $V = \mathbb{R}^n$.
where $W = \{(a_1, a_2, \cdots a_n) / a_1 \geq 0\}$.

Sol^n^ If $a_1 = 3$ then $a_1 > 0$

$$\alpha = (3, a_2, a_3, \cdots a_n)$$

If $a = -2 \in \mathbb{R}$

then $a\alpha = (-6, -2a_2, -2a_3, \cdots, -2a_n) \notin W$

$\therefore W$ is not a subspace of $\mathbb{R}^n$. $(\because a_1 = -6 < 0)$

→ S.T W is not a subspace of $\mathbb{R}^n$.
where $W = \{(a_1, a_2 \cdots a_n) / a_2 = a_1^2\} \subseteq \mathbb{R}^n$

Let $a \in \mathbb{R}$; $\alpha = (a_1, a_2, \cdots a_n) \in W$ and $a_2 = a_1^2$.

$\Rightarrow a\alpha$ need not be an elt of W.

for example
let $a = \frac{1}{2} \in \mathbb{R}$, $\alpha = (2, 4, a_3, \cdots a_n) \in W$

$$\Rightarrow a\alpha = \left(1, 2, \frac{a_3}{2}, \cdots \frac{a_n}{2}\right) \notin W$$

$(\because 2 \neq 1^2$ i.e. $a_2 \neq a_1^2)$

→ Let V be the real vector space of all functions f from $\mathbb{R}$ into $\mathbb{R}$.
which of the following sets of functions are subspaces of V.

(i) $W = \{f / f(3) = 0\}$
(ii) $W = \{f / f(7) = f(1)\}$
(iii) $W = \{f / f(-x) = -f(x)\}$
(iv) $W = \{f / f(7) = 2 + f(1)\}$

(v) $W = \{f \mid f(x) = [f(x)]^2\}$

(vi) W consists of the continuous functions

(vii) W consists of the differentiable functions.

Sol^n: (i) Let $a, b \in \mathbb{R}$; $f, g \in W$ s.t. $f(3) = 0$ & $g(3) = 0$

$$\Rightarrow (af+bg)(3) = (af)(3) + (bg)(3)$$
$$= a f(3) + b g(3)$$
$$= a(0) + b(0)$$
$$= 0$$

$\therefore af+bg \in W$

$\therefore W$ is a subspace of V.

(ii) $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(7) = f(1)$ and $g(7) = g(1)$

$$\Rightarrow (af+bg)(7) = (af)(7) + (bg)(7)$$
$$= a f(7) + b g(7)$$
$$= a f(1) + b g(1)$$
$$= (af)(1) + (bg)(1)$$
$$= (af+bg)(1)$$

$\therefore af+bg \in W$

$\therefore W$ is a subspace of V.

(iii) $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(-x) = -f(x)$ & $g(-x) = -g(x)$

$$\Rightarrow (af+bg)(-x) = (af)(-x) + (bg)(-x)$$
$$= a f(-x) + b g(-x)$$
$$= a[-f(x)] + b[-g(x)]$$
$$= -[a f(x) + b g(x)]$$
$$= -[(af)(x) + (bg)(x)]$$
$$= -(af+bg)(x).$$

$\therefore af+bg \in W.$

$\therefore W$ is a subspace of V.

(iv) $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(7) = 2 + f(1)$ &
$g(7) = 2 + g(1)$

$$\Rightarrow (af + bg)(7) = (af)(7) + (bg)(7)$$
$$= a f(7) + b\, g(7)$$
$$= a[2 + f(1)] + b(2 + g(1))$$
$$= 2a + a f(1) + 2b + b\, g(1)$$
$$= (2a + 2b) + (af + bg)(1) \quad \text{--- (1)}$$

Let $a = 1$, $b = 1$ then

$$(f+g)(7) = 4 + (f+g)(1)$$
$$\neq 2 + (f+g)(1)$$

$\therefore f + g \notin W$
$\therefore W$ is not a subspace

(v) Let $a, b \in \mathbb{R}$; $f, g \in W$ i.e, $f(x^2) = [f(x)]^2$ &
$g(x^2) = [g(x)]^2$

$$\Rightarrow (af + bg)(x^2) = a f(x^2) + b\, g(x^2)$$
$$= a[f(x)]^2 + b[g(x)]^2 \quad \text{--- (1)}$$

Now $(af + bg)(x^2) = [(af + bg)(x)]^2$
$$= [a f(x) + b\, g(x)]^2$$
$$= a^2[f(x)]^2 + b^2[g(x)]^2 + 2ab\, f(x)\, g(x) \quad \text{--- (2)}$$

$\therefore$ from (1) & (2)

$$a[f(x)]^2 + b[g(x)]^2 \neq a^2[f(x)]^2 + b^2[g(x)]^2 + 2ab\, f(x)\, g(x)$$

$\therefore af + bg \notin W$

$\therefore W$ is not a subspace

(vi) If f and g are continuous functions and $a, b \in \mathbb{R}$ then $af + bg$ is also continuous function.

$\therefore af+bg \in W$

$\therefore$ W is a subspace of V.

(vi) If f and g are differentiable functions and $a, b \in \mathbb{R}$ then

$af+bg$ is also differentiable.

$\therefore$ W is a subspace of V.

→ Let $W = \{(x_1, x_2 \cdots x_n) \in V_n / x_1 = 0\}$. Prove that W is a subspace of V_n

→ Prove that $W = \{(x_1, x_2 \cdots x_n) \in V_n^C / \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = 0,$ α_i's are given constants$\}$

where V_n^C — the set of all ordered n-tuples of complex numbers.

is a subspace of V_n^C.

→ Which of the following sets are subspaces of V_3?

(a) $\{(x_1, x_2, x_3) / x_1 x_2 = 0\}$ (b) $\{(x_1, x_2, x_3) / \frac{x_2}{x_1} = \sqrt{2}\}$

(c) $\{(x_1, x_2, x_3) / \sqrt{2} x_1 = \sqrt{3} x_2\}$ (d) $\{(x_1, x_2, x_3) / x_3$ is an integer$\}$

(e) $\{(x_1, x_2, x_3) / x_1^2 + x_2^2 + x_3^2 \leq 1\}$ (f) $\{(x_1, x_2, x_3) / x_1 + x_2 + x_3 \geq 0\}$

(g) $\{(x_1, x_2, x_3) / x_1 = \sqrt{2} x_2$ and $x_3 = 3x_2\}$

(h) $\{(x_1, x_2, x_3) / x_1 - 2x_2 = x_3 - \frac{3x_2}{2}\}$.

(i) $\{(x_1, x_2, x_3) / x_1 = 2x_2$ or $x_3 = 3x_2\}$.

Ans: (c), (g) & (h) are subspaces of V_3.

→ Which of the following sets are subspaces of $\mathcal{P}$?

(a) $\{p \in \mathcal{P} /$ degree of $p = 4\}$. (b) $\{p \in \mathcal{P} /$ degree of $p \leq 3\}$

(c) $\{p \in \mathcal{P} /$ degree of $p \geq 5\}$ (d) $\{p \in \mathcal{P} /$ degree of $p \leq 4$ and $p'(0) = 0\}$

(e) $\{p \in \mathcal{P} / p(1) = 0\}$.

Ans: (b), (d) & (e) are subspaces of $\mathcal{P}$.

→ Which of the following sets are subspaces of $\mathcal{C}[a,b]$?

(a) $\{f \in \mathcal{C}(a,b) / f(x_0) = 0, x_0 \in (a,b)\}$

(b) $\{f \in \mathcal{C}(a,b) / f'(x) = 0$ for all $x \in (a,b)\}$

(c) $\{f \in \mathcal{C}(a,b) / f\left(\frac{a+b}{2}\right) = 1\}$

(d) $\{f \in \mathcal{C}(a,b) / f(x) = x^2 f(x)\}$

(e) $\{f \in \mathcal{C}(a,b) / 2f'''(x) + 3x f''(x) - f'(x) + x^2 f(x) = 0\}$

(f) $\{f \in \mathcal{C}(a,b) / \int_a^b f(x)\,dx = 0\}$

Ans: (a), (b), (d), (e) and (f) are subspaces of $\mathcal{C}(R)$

→ $\mathcal{C}[a,b]$ is a subspace of $\mathcal{F}[a,b]$.

because the sum of two continuous functions is continuous and any scalar multiple of a continuous function is again continuous, we find that addition and scalar multiplication are closed in $\mathcal{C}[a,b]$.

This observation not only proves that $\mathcal{C}[a,b]$ is a vector space, but also that it is a subspace of $\mathcal{F}[a,b]$.

Note: The spaces $\mathcal{C}[a,b]$, $\mathcal{C}^{(1)}[a,b]$, $\mathcal{C}^{(n)}[a,b]$ and $\mathcal{P}[a,b]$ are subspaces of $\mathcal{F}[a,b]$.

Further, note that

(a) $\mathcal{P}[a,b]$ is a subspace of $\mathcal{C}[a,b]$

(b) $\mathcal{C}^{(1)}[a,b]$ is a subspace of $\mathcal{C}[a,b]$.

(c) $\mathcal{C}^{(n)}[a,b]$ is a subspace of $\mathcal{C}[a,b]$ for every positive integer n.

(d) $\mathcal{C}^{(n)}[a,b]$ is a subspace of $\mathcal{C}^{(m)}[a,b]$ for every $m < n$.

(e) $\mathcal{P}[a,b]$ is a subspace of $\mathcal{C}^{(n)}[a,b]$ for every positive integer n.

(f) Similar results are true for functions defined on (a,b).

→ Let V be the vector space of all real sequences $\langle \alpha_n \rangle$.

(i) prove that $W = \{\langle a_n \rangle \in V : \lim_{n\to\infty} a_n = 0\}$ is a subspace of V.

(ii) prove that $U = \{\langle a_n \rangle \in V : \sum_{n=1}^{\infty} a_n^2$ is finite$\}$ is a subspace of V and is contained in W.

Soln: Let $\alpha, \beta \in \mathbb{R}$ and $\langle a_n \rangle \in W$, $\langle b_n \rangle \in W$.

$\therefore \lim_{n\to\infty} a_n = 0$ & $\lim_{n\to\infty} b_n = 0$

Now $\alpha \langle a_n \rangle + \beta \langle b_n \rangle = \langle \alpha a_n + \beta b_n \rangle$.

where $\lim_{n\to\infty} (\alpha a_n + \beta b_n) = \alpha \lim_{n\to\infty} a_n + \beta \lim_{n\to\infty} b_n$

$= \alpha . 0 + \beta . 0$

$= 0$.

$\therefore \alpha \langle a_n \rangle + \beta \langle b_n \rangle \in W$.

$\therefore W$ is a subspace of V.

(ii) Let $\alpha, \beta \in \mathbb{R}$ and $\langle a_n \rangle \in U$, $\langle b_n \rangle \in U$.

$\therefore \sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ are finite.

i.e, each one of them is a convergent series.

It follows that $\alpha \sum_{n=1}^{\infty} a_n^2 + \beta \sum_{n=1}^{\infty} b_n^2$ is finite

i.e, $\alpha \langle a_n \rangle + \beta \langle b_n \rangle \in U$.

Hence U is a subspace of V.

Let $\langle a_n \rangle \in U$ be arbitrary

Then $\sum_{n=1}^{\infty} a_n^2$ is a convergent series and

So $\lim_{n\to\infty} a_n^2 = 0$

$\Rightarrow \lim_{n\to\infty} a_n \lim_{n\to\infty} a_n = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$

$\Rightarrow \langle a_n \rangle \in W$

$\therefore$ Hence $U \subseteq W$.

→ Let V be the vector space of all 2×2 matrices over the field $\mathbb{R}$ of real numbers.

Let (i) $W_1 = \{ A \in V / A^2 = A \}$

(ii) $W_2 = \{ A \in V / \det A = 0 \}$

Show that W_1 and W_2 are not subspaces of V.

→ Show that W is a subspace of V where W consists of all matrices which commute with a given matrix T; that is, $W = \{ A \in V / AT = TA \}$.

Sol^n: Given ~~that~~ W consists of ~~all matrices~~ which commute with a given matrix

i.e, $W = \{ A \in V / AT = TA \}$

Since $OT = O = TO$

$\therefore O \in W$.

$\therefore W$ is non-empty.

Now suppose $A, B \in W$

i.e, $AT = TA$ and $BT = TB$

For any scalars $a, b \in F$,

$$(aA + bB)T = (aA)T + (bB)T$$
$$= a(AT) + b(BT)$$

$$= a(TA) + b(TB)$$
$$= T(aA) + T(bB)$$
$$= T(aA + bB).$$

Thus $aA + bB$ commutes with T.

$\Rightarrow aA + bB \in W$.

Hence W is a subspace of V.

→ Show that W is a subspace of V; where W consists of the bounded functions

[A function $f \in V$ is bounded if there exists $M > 0$ ($M \in \mathbb{R}$) such that $|f(x)| \leq M$ for every $x \in \mathbb{R}$]

Sol: Since $0(x) = 0$ for every $x \in \mathbb{R}$.

clearly 0 is bounded.

$\therefore 0 \in W$

i.e, W is non empty.

Now let $f, g \in W$ with M_f and M_g bounds for f and g respectively. i.e, $|f(x)| \leq M_f$ & $|g(x)| \leq M_g$.

Then for any scalars $a, b \in \mathbb{R}$ and $\forall x \in \mathbb{R}$

$$|(af + bg)(x)| = |a f(x) + b g(x)|$$
$$\leq |a f(x)| + |b g(x)|$$
$$= |a| \, |f(x)| + |b| \, |g(x)|.$$
$$\leq |a| M_f + |b| M_g.$$

$\Rightarrow |a| M_f + |b| M_g$ is a bound for the function $af + bg$.

Thus W is a subspace of V.

→ which of the following sets of vectors $\alpha = (a_1, a_2 \ldots a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geq 3)$?

(a) all α such that $a_1 \geq 0$.

(b) all α such that $a_1 + 3a_2 = a_3$.

(c) all α such that $a_2 = a_1^2$

(d) all α such that $a_1 a_2 = 0$

(e) all α such that a_2 is rational.

Linear Combination:

Defn: Let $V(F)$ be a vector space. $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ then any vector $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$ where $a_1, a_2 \ldots a_n \in F$ is called a linear combination of the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$.

Linear Span:

Let $V(F)$ be a vector space. $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$. Then the collection of all linear combinations of a finite number of elements of 'S' is called linear span of S and is denoted by $L(S)$.

i.e, $L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n / \alpha_1, \alpha_2, \ldots \alpha_n \in S, \; a_1, a_2, \ldots a_n \in F\}$.

Smallest subspace containing any subset of V(F)S

Defn: Let V(F) be a vector space and S be any subset of V (i.e, $S \subseteq V$). If U is a subspace of V containing S and U is contained in every subspace of V containing 'S' then U is called the smallest subspace of V containing S.

$S \subseteq V$
If $U \subseteq V$ and $U \subseteq V$

→ The smallest subspace of V containing S is also called the subspace of V generated or spanned by S and is denoted by {S}. i.e, {S} = U

→ If {S} = V then we say that V is spanned by S.

Theorem:

2003 If V(F) is a vector space, $S \subseteq V$, then the linear span of S is the smallest subspace of V(F) containing S.

(i.e, L(S) is a subspace of V(F) generated by S i.e, L(S) = {S}.)

proof: Given that V(F) is a vector space and $S \subseteq V$.

Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and $L(S) = \{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n \mid \alpha_1, \alpha_2, \ldots \alpha_n \in S,\ a_1, a_2, \ldots a_n \in F\} \subseteq V$

Now $\forall a, b \in F$; $\alpha, \beta \in L(S)$

Choose $\alpha = a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n$

$\beta = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n$

where a's, b's $\in F$ and α's $\in S$

$\Rightarrow a\alpha + b\beta = a(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$
$\qquad + b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$
$\qquad = (aa_1 + bb_1)\alpha_1 + (aa_2 + bb_2)\alpha_2 + \cdots + (aa_n + bb_n)\alpha_n$
$\qquad \in L(S)$
$\qquad (\because aa_1 + bb_1, aa_2 + bb_2, \ldots \ldots aa_n + bb_n \in F)$

$\therefore L(S)$ is a subspace of $V(F)$.

Let $\alpha_i \in S$; $i = 1, 2, \ldots n$

then $\alpha_i = 1\alpha_i$
$\quad$ = Linear combination of α_i
$\quad \in L(S)$

$\therefore \alpha_i \in L(S)$

$\therefore S \subseteq L(S)$

Now let W be any subspace of $V(F)$ containing S.
$\therefore S \subseteq W$.

if $\alpha \in L(S)$ then α = the linear combination of a finite no. of elts of S.

$\qquad \in W \quad (\because S \subseteq W)$

$\therefore$ If $\alpha \in L(S)$ then $\alpha \in W$

$\therefore L(S) \subseteq W$.

$\therefore S \subseteq L(S) \subseteq W \subseteq V$.

$\therefore L(S)$ ~~is~~ the smallest subspace of V containing S.

i.e., $L(S) = \{S\}$.

Note: If in any case, we are to prove that $L(S) = V$ then we are enough to prove that $V \subseteq L(S)$ because w.k.T $L(S) \subseteq V$ ($\because L(S)$ is a subspace of V)

In order to prove that $V \subseteq L(S)$

for this each elt of V can be expressed as linear combination of a finite no. of elts of S.

$\therefore$ Each elt of V will also the elt of $L(S)$.

i.e, let $\alpha \in V \Rightarrow \alpha =$ the l.c. of finite no. of elts of S.

$$\in L(S)$$

$$\therefore \alpha \in L(S)$$

$$\therefore V \subseteq L(S).$$

$$\therefore V \subseteq L(S) \text{ and } L(S) \subseteq V$$

$$\Rightarrow L(S) = V.$$

Ex The subset $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ of $V_3(F)$. (i.e, $S \subseteq V_3(F)$) generates or spans the entire vector space $V_3(F)$ i.e, $L(S) = V_3$

Sol^n W.K.T. $L(S) \subseteq V_3$ —— ①

Let $\alpha = (a,b,c) \in V_3$ then

$$\alpha = (a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$$

$$\in L(S)$$

$$\therefore \alpha \in L(S)$$

$$\therefore V_3 \subseteq L(S). \text{ —— (2)}$$

$\therefore$ from (1) & (2) we have $L(S) = V_3$.

Defn Linear sum of two subspaces

Let W_1 & W_2 be any two subspaces of $V(F)$ then the set $\{\alpha_i + \alpha_j / \alpha_i \in W_1 \text{ \& } \alpha_j \in W_2\} \subseteq V$ is called linear sum of W_1 & W_2 and is denoted by $W_1 + W_2$.

i.e, $W_1 + W_2 = \{\alpha_i + \alpha_j / \alpha_i \in W_1, \alpha_j \in W_2\} \subseteq V$.

Theorem: Let W_1 and W_2 be two subspaces of $V(F)$, then the linear sum $W_1 + W_2$ is a subspace of $V(F)$ and $W_1 + W_2 = L(W_1 \cup W_2)$.

i.e, $W_1 + W_2 = \{W_1 \cup W_2\}$.

Imp

proof: Given that

$V(F)$ is a vectorspace.

W_1 & W_2 are two subspaces of $V(F)$.

$W_1+W_2 = \{\alpha_i + \alpha_j / \alpha_i \in W_1, \alpha_j \in W_2\} \subseteq V$.

Let $a, b \in F$; $\alpha, \beta \in W_1+W_2$

Choose $\alpha = \alpha_i + \alpha_j$; $\alpha_i \in W_1$, $\alpha_j \in W_2$

$\beta = \alpha_k + \alpha_l$; $\alpha_k \in W_1$, $\alpha_l \in W_2$

$\Rightarrow a\alpha + b\beta = a(\alpha_i + \alpha_j) + b(\alpha_k + \alpha_l)$

$= (a\alpha_i + b\alpha_k) + (a\alpha_j + b\alpha_l)$

$\in W_1 + W_2$.

Since W_1 is a subspace $\therefore a\alpha_i + b\alpha_k \in W_1$

and W_2 is a subspace $\therefore a\alpha_j + b\alpha_l \in W_2$

$\therefore W_1+W_2$ is a subspace of $V(F)$.

Now $0 \in W_1$, $x \in W_2 \Rightarrow 0 + x \in W_1 + W_2$

$\Rightarrow x \in W_1 + W_2$

$\therefore W_2 \subseteq W_1 + W_2$ —— (1)

$y \in W_1$, $0 \in W_2 \Rightarrow y + 0 \in W_1 + W_2$

$\Rightarrow y \in W_1 + W_2$

$\therefore W_1 \subseteq W_1 + W_2$ —— (2)

$\therefore$ from (1) & (2) we have

$W_1 \cup W_2 \subseteq W_1 + W_2 \subseteq V$

W.K.T Linear Spans of $W_1 \cup W_2$ (i.e, $L(W_1 \cup W_2)$)

is the smallest subspace of $V(F)$ containing $W_1 \cup W_2$

$\therefore L(W_1 \cup W_2) \subseteq W_1 + W_2$ —— (3)

Let $\alpha \in W_1 + W_2 \Rightarrow \alpha = \alpha_i + \alpha_j \in W_1 + W_2$

Now $\alpha_i \in W_1$, $\alpha_j \in W_2 \Rightarrow \alpha_i, \alpha_j \in W_1 \cup W_2$

Now $\alpha \in W_1 + W_2 \Rightarrow \alpha = \alpha_i + \alpha_j$

$= 1\alpha_i + 1\alpha_j$

= l.c. of finite no. of elts of $W_1 \cup W_2$.
$\in L(W_1 \cup W_2)$

$\therefore W_1 + W_2 \subseteq L(W_1 \cup W_2)$ —— (4)

$\therefore$ from (3) & (4)
we have $L(W_1 \cup W_2) = W_1 + W_2$.

→ If: S, T are subsets of V(F) then

(i) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$.

(ii) $L(S \cup T) = L(S) + L(T)$

(iii) $L(L(S)) = L(S)$.

proof. Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ then
any vector $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(S)$

Since $S \subseteq T$
$\Rightarrow S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq T$.
$\therefore \alpha \in L(T)$.

$\therefore$ If $\alpha \in L(S)$ then $\alpha \in L(T)$.
$\therefore L(S) \subseteq L(T)$.

(ii) Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$
and $T = \{\beta_1, \beta_2, \ldots \beta_p\} \subseteq V$
then $S \cup T = \{\alpha_1, \alpha_2 \ldots \alpha_n, \beta_1, \beta_2, \ldots \beta_p\} \subseteq V$.

Let $\alpha \in L(S \cup T)$ then
$\therefore \alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p$

Since $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \in L(S)$
and $b_1\beta_1 + b_2\beta_2 + \cdots + b_p\beta_p \in L(T)$.

$\therefore \alpha \in L(S) + L(T)$

$\therefore L(S \cup T) \subseteq L(S) + L(T)$ —— (1)

Let $v \in L(S) + L(T)$ then $v = \beta + \delta$.
where $\beta \in L(S)$ & $\delta \in L(T)$.

Now β = L.C. of finite no. of elts of S and
δ = L.C of finite no of elts of T.
$\therefore \beta+\delta$ = L.C. of finite no. of elts of $S \cup T$.
$\therefore \gamma = \beta+\delta \in L(S \cup T)$
$\therefore$ If $\gamma \in L(S)+L(T)$ then
$\gamma \in L(S \cup T)$
$\therefore L(S)+L(T) \subseteq L(S \cup T)$ — (2)

$\therefore$ from (1) & (2) we have
$L(S \cup T) = L(S)+L(T)$.

(iii) $L(L(S))$ is the smallest subspace of V containing $L(S)$.
But $L(S)$ is a subspace of V.
$\therefore$ the smallest subspace of V containing $L(S)$ is $L(S)$ itself.
i.e, $L(S) \subseteq L(L(S)) \subseteq L(S) \subseteq V$
$\therefore L(L(S)) = L(S)$

$L(S) \subseteq L(L(S)) \subseteq W \subseteq V$, $W = L(S)$

Defn: Linear dependence of vectors:

$V(F)$ is a vector space and $S = \{\alpha_1, \alpha_2, \dots \alpha_n\} \subseteq V$

If $\exists$ atleast one non-zero scalar $a_1, a_2, \dots a_n \in F$

such that $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$

Then S is called linear dependent.

Linear Independence of vectors:

$V(F)$ is a vector space and $S = \{\alpha_1, \alpha_2, \dots \alpha_n\} \subseteq V$

If $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$; $a_i \in F$, $1 \le i \le n$

$\Rightarrow a_1 = a_2 = \dots = a_n = 0$

i.e., $a_i = 0$ for each $1 \le i \le n$

Ex: $V_n(F) = \{(a_1, a_2 \dots a_n) / a_1, a_2, \dots a_n \in F\}$ is a vector space.

$S = \{(1, 0, 0, \dots 0), (0, 1, 0, \dots 0), (0, 0, 1, 0, \dots 0) \dots (0, 0, \dots 0, 1)\} \subseteq V_n(F)$.

Now $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$

$\Rightarrow a_1(1, 0, 0, \dots 0) + a_2(0, 1, 0, \dots 0) + \dots + a_n(0, 0, \dots 1) = (0, 0, \dots 0)$

$\Rightarrow (a_1, 0, \dots 0) + (0, a_2, \dots 0) + \dots + (0, 0, \dots a_n) = (0, 0, \dots 0)$

$\Rightarrow (a_1, a_2, \dots a_n) = (0, 0, \dots 0)$

$\Rightarrow a_1 = a_2 = \dots = a_n = 0$

$\therefore S$ is L.I.

Ex S.T $S = \{(1,2,1), (3,1,5), (3,-4,7)\} \subseteq \mathbb{R}^3$ is L.D.

Sol^n – $\forall\, a, b, c \in \mathbb{R}$

$$a(1,2,1) + b(3,1,5) + c(3,-4,7) = (0,0,0)$$

$$\Rightarrow (a, 2a, a) + (3b, b, 5b) + (3c, -4c, 7c) = (0,0,0)$$

$$\Rightarrow (a+3b+3c,\ 2a+b-4c,\ a+5b+7c) = (0,0,0)$$

$$\Rightarrow a+3b+3c = 0 \quad \text{--- (1)}$$

$$2a+b-4c = 0 \quad \text{--- (2)}$$

$$a+5b+7c = 0 \quad \text{--- (3)}$$

Solving these equations, we get

$(1)-(3) \equiv -2b-4c = 0$

$$\Rightarrow b = -2c \quad \text{--- (4)}$$

$\therefore (1) \equiv a - 6c + 3c = 0$

$$\Rightarrow a - 3c = 0$$

$$\Rightarrow a = 3c \quad \text{--- (5)}$$

Substituting (4) & (5) in (2) we get

$$6c - 2c - 4c = 0$$

$$\Rightarrow 0 = 0$$

$\therefore \exists$ non-zero values for a, b, c to satisfy the equations (1) (2) & (3)

$\therefore$ The given set is L.D.

Theorem -

→ If two vectors are linear dependent then one of them is a scalar multiple of the other.

Sol^n Let α, β be two linear dependent vectors of the vector space $V(F)$.

$\therefore$ $\exists$ atleast one of the scalar $a, b \in F$ is non-zero s.t $a\alpha + b\beta = 0$

if $a \neq 0$ then $a\alpha = -b\beta$.

$$\Rightarrow \alpha = \left(\frac{-b}{a}\right)\beta.$$

$\therefore$ α is scalar multiple of β.

If $b \neq 0$ then $b\beta = -a\alpha$

$$\Rightarrow \beta = \left(-\frac{a}{b}\right)\alpha$$

$\therefore$ β is the scalar multiple of α.

$\therefore$ One of the vectors α and β is scalar multiple of the other.

Theorem A set consisting of single non-zero vector is always L.I.

proof Let $V(F)$ be a vector space.

$S = \{\alpha\} \subseteq V$; $\alpha \neq 0$

if $a \in F$ then $a\alpha = 0$

$\Rightarrow a = 0$ $(\because \alpha \neq 0)$

$\therefore$ S is L.I.

Theorem: If the set $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ of vectors of $V(F)$ is L.I then none of the vectors $\alpha_1, \alpha_2, \cdots \alpha_n$ can be zero vector.

Proof: Given that

$S = \{\alpha_1, \alpha_2, \dots \alpha_n\} \subseteq V$ is L.I

$\therefore$ $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$, $a_1, a_2, \dots a_n \in F$
$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0$

If possible let $\alpha_K = 0$; $1 \le K \le n$.

then $0\alpha_1 + 0\alpha_2 + \cdots + a_K\alpha_K + 0\alpha_{K+1} + \cdots + 0\alpha_n = 0$

for any $a_K \neq 0$ in F.

Since $a_K \neq 0$

$\therefore$ S is LD.

which is contradiction to the hypothesis that S is L.I.

Our assumption that $\alpha_K = 0$; $1 \le K \le n$ is wrong.

$\therefore$ None of the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ can be zero vector.

Theorem → A set of vectors which containing the zero vector is LD.

proof Let V(F) be the vector space.

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and $\alpha_K = \bar{0}$; $1 \le K \le n$.

Consider linear combination

$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_K\alpha_K + a_{K+1}\alpha_{K+1} + \cdots + a_n\alpha_n = 0$

Taking $a_1 = a_2 = \cdots = a_{K+1} = \cdots = a_n = 0$ and $a_K \neq 0$

$\therefore 0\alpha_1 + 0\alpha_2 + \cdots + a_K\alpha_K + 0\alpha_{K+1} + \cdots + 0\alpha_n = 0$

$\Rightarrow a_K\alpha_K = 0$

$\Rightarrow a_K \neq 0$ $(\because \alpha_K = 0)$

$\therefore$ S is L.D.

Theorem A subset of a LI set is L.I.

proof V(F) is a vector space

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$ is LI

Now let $S' = \{\alpha_1, \alpha_2, \ldots \alpha_K\} \subseteq S$ $(1 \le K \le n)$

then $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_K\alpha_K = 0$; $a_1, a_2, \ldots a_K \in F$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_k\alpha_k + 0\alpha_{k+1} + 0\alpha_{k+2} + \cdots\cdots + 0\alpha_n = 0$

$\Rightarrow a_1 = a_2 = \cdots\cdots = a_k = 0$ $(\because S$ is LI$)$

$\therefore S'$ is LI.

Theorem A superset of a linear dependent set of vectors is LD.

proof Let $V(F)$ be a vector space.

and $S = \{\alpha_1, \alpha_2, \cdots\cdots \alpha_n\} \subseteq V$ is LD.

Now let $S' = \{\alpha_1, \alpha_2, \cdots\cdots \alpha_n, \beta_1, \beta_2, \cdots \beta_k\} \supseteq S$.

Since S is LD

$\therefore \exists$ atleast one of the scalar $a_1, a_2, \cdots a_n \in F$ is not zero. s.t

$a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_n\alpha_n = 0$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_n\alpha_n + 0\beta_1 + 0\beta_2 + \cdots\cdots + 0\beta_k = 0$

Since in the above relation the scalar coefficients not all zero.

$\therefore S'$ is LD.

Theorem Let $V(F)$ be vector space. and $S = \{\alpha_1, \alpha_2 \cdots \alpha_n\} \subseteq V$ (contains non-zero vectors) if S is LD then one of the vectors of S say α_i $(1 < i \leq n)$ is a linear combination of its preceding vectors.

proof $V(F)$ is a vectorspace

$S = \{\alpha_1, \alpha_2 \cdots \alpha_n\} \subseteq V$

and S contains non-zero vectors.

Since S is LD.

$\therefore \exists$ atleast one scalar $a_1, a_2, \cdots a_n \in F$ is non-zero s.t $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k + \cdots + a_n\alpha_n = 0$ (1)

Suppose that the maximum value of k for which $a_k \neq 0$ is 'i'.

i.e, $a_i \neq 0$ and $a_{i+1} = a_{i+2} = \cdots = a_n = 0$

if this maximum value is one then $a_1 \neq 0$

and $a_2 = a_3 = \cdots = a_n = 0$

$(1) \equiv a_1\alpha_1 + 0\alpha_2 + \cdots + 0\alpha_n = 0$

$\Rightarrow a_1\alpha_1 = 0$

$\Rightarrow \alpha_1 = 0 \quad (\because a_1 \neq 0)$

which is contradiction to the hypothesis that S contains non-zero vectors.

$\therefore i \neq 1$.

$\therefore 1 < i \leq n$

$(1) \equiv a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + \cdots + a_{i-1}\alpha_{i-1} + a_i\alpha_i + 0\alpha_{i+1} + 0\alpha_{i+2} + \cdots + 0\alpha_n = 0$

$\Rightarrow a_i\alpha_i = -a_1\alpha_1 - a_2\alpha_2 - \cdots - a_{i-1}\alpha_{i-1}$

$\Rightarrow \alpha_i = \left(\frac{-a_1}{a_i}\right)\alpha_1 + \left(\frac{-a_2}{a_i}\right)\alpha_2 + \cdots + \left(\frac{-a_{i-1}}{a_i}\right)\alpha_{i-1}$

$\therefore \alpha_i \, (1 < i \leq n)$ is a linear combination of its preceding vectors.

→ Let $V(F)$ be the vector space. $S = \{\alpha_1, \alpha_2, \cdots \alpha_n\} \subseteq V$ (contains non-zero vectors)

If one of the vectors of 'S' say $\alpha_i \, (1 < i \leq n)$ is a linear combination of its preceding vectors then S is L.D.

Proof: Given that

$V(F)$ is a vector space.

$S = \{\alpha_1, \alpha_2, \cdots \alpha_n\} \subseteq V$

and one of the vectors of S say $\alpha_i \; (1 < i \leq n)$ is a linear combination of its preceding vectors.

$\therefore \alpha_i = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + (-1)\alpha_i = 0$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + (-1)\alpha_i + 0\alpha_{i+1} + 0\alpha_{i+2}$
$+ \cdots + 0\alpha_n = 0$

$\therefore$ Coefficient of $\alpha_i = -1 \neq 0$

$\therefore S$ is LD.

Theorem: Let $V(F)$ be a vector space. $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$. If one of the vectors of S is a linear combination of all the remaining vectors then S is LD.

proof: $V(F)$ is a vector space

$S = \{\alpha_1, \alpha_2, \ldots \alpha_n\} \subseteq V$

and one of the vectors of S is a linear combination of all the remaining vectors.

$\therefore \alpha_i = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1} + a_{i+1}\alpha_{i+1}$
$+ \cdots + a_n\alpha_n$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + (-1)\alpha_i + a_{i+1}\alpha_{i+1} + a_{i+2}\alpha_{i+2}$
$+ \cdots + a_n\alpha_n = 0$

$\therefore$ The coefficient of $\alpha_i \neq 0$.

$\therefore S$ is LD.

Theorem: If in a vector space $V(F)$, a vector β is a linear combination of the set of vectors $\alpha_1, \alpha_2 \ldots \alpha_n$ then the set of vectors $\beta, \alpha_1, \alpha_2 \ldots \alpha_n$ is L.D.

Sol: Since β is a linear combination of $\alpha_1, \alpha_2 \ldots \alpha_n$

$\therefore \exists$ scalars $a_1, a_2, \ldots a_n \in F$ s.t

$$\beta = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n + (-1)\beta = 0$$

$\therefore$ In the above relation the coefficient of $\beta = -1 \neq 0$

$\therefore$ In the above relation not all the scalar coefficients are zero.

$\therefore$ The set of vectors $\alpha_1, \alpha_2, \cdots \alpha_n, \beta$ is LD.

→ Write the vector $\alpha = (1,-2,5)$ as a linear combination of the elements of the set $\{(1,1,1), (1,2,3), (2,-1,1)\} \subseteq \mathbb{R}^3$.

Sol^n: $\alpha = (1,-2,5) = a(1,1,1) + b(1,2,3) + c(2,-1,1)$

$$= (a+b+2c,\ a+2b-c,\ a+3b+c)$$

$$\Rightarrow a+b+2c = 1 \quad \text{——— (1)}$$
$$a+2b-c = -2 \quad \text{——— (2)}$$
$$a+3b+c = 5 \quad \text{——— (3)}$$

$(1)-(2) = -b+3c = 3$ ——— (4)

$(2)-(3) = -b-2c = -7$ ——— (5)

$(4)-(5) = 5c = 10$

$\Rightarrow \boxed{c=2}$

(4) $\equiv -b = -3$

$\Rightarrow b = 3$

(1) $\equiv a+3+4 = 1$

$\Rightarrow \boxed{a=-6}$

$\therefore (1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$

H.W → Express $\alpha = (2,-5,3)$ in $\mathbb{R}^3$ as a linear combination of the vectors $e_1 = (1,-3,2)$, $e_2 = (2,-4,-1)$ and $e_3 = (1,-5,7)$

→ Express the polynomial $\alpha = (t^2+4t-3)$ as a linear combination of the polynomials $e_1 = t^2-2t+5$, $e_2 = 2t^2-3t$ and $e_3 = t+3$.

Sol: $\alpha = a e_1 + b e_2 + c e_3$; where a, b, c are unknown scalars. —— (I)

$\Rightarrow t^2+4t-3 = a(t^2-2t+5) + b(2t^2-3t) + c(t+3)$

$= (a+2b)t^2 + (-2a-3b+c)t + (5a+3c)$.

$\Rightarrow a+2b = 1$ —— (1)

$-2a-3b+c = 4$ —— (2)

$5a+3c = -3$ —— (3)

$2\times(2) + 3\times(1) \equiv -a+2c = 11$ —— (4)

$(3) + 5\times(4) \equiv 13c = 52$ $\Rightarrow \boxed{c=4}$; $(4) \equiv -a = 3$ $\Rightarrow \boxed{a=-3}$

$(1) \equiv -3+2b = 1$

$\Rightarrow \boxed{b=2}$

$\therefore (I) \equiv \alpha = -3e_1 + 2e_2 + 4e_3$.

$\Rightarrow$ Write the matrix $E = \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix}$ as a linear combination of the matrices $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$

Sol: $E = xA + yB + zC$ —— (I) where x, y, z are unknown scalars.

$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = x\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 2 \\ 0 & -1 \end{pmatrix}$

$= \begin{pmatrix} x & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y & y \end{pmatrix} + \begin{pmatrix} 0 & 2z \\ 0 & -z \end{pmatrix}$

$\begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} x & x+2z \\ x+y & y-z \end{pmatrix}$

$\therefore \boxed{x=3}$

$x+2z = 1$; $x+y = 1$

$\Rightarrow \boxed{z=-1}$ $\Rightarrow \boxed{y=-2}$

$\therefore (I) \equiv E = 3A - 2B + (-1)\cdot C$

→ Determine whether α & β are L.D.
where (a) $\alpha = (3,4)$, $\beta = (1,-3)$
(b) $\alpha = (2,-3)$, $\beta = (6,-9)$

Sol^n (a) Since no vector is a scalar multiple of the other.
$\therefore$ α & β are not L.D.

b). Since β is a scalar multiple of α.
i.e, $(6,-9) = 3(2,-3)$
i.e, $\beta = 3\alpha$.
$\therefore$ α & β are L.D. vectors

→ Determine whether α & β are LD
where (a) $\alpha = (4,3,-2)$, $\beta = (2,-6,7)$
(b) $\alpha = (-4,6,-2)$, $\beta = (2,-3,1)$

Sol^n a) neither is a scalar multiple of the other.
$\therefore$ α and β are not LD.

b) $\alpha = (-2)\beta$.
$\therefore$ α and β are LD.

→ S.T $S = \{(1,2,4), (1,0,0), (0,1,0), (0,0,1)\}$ is a LD subset of $V_3(R)$.

Sol^n Since one of the vector of S is a linear combination of all the remaining vectors.
i.e, $(1,2,4) = 1(1,0,0) + 2(0,1,0) + 4(0,0,1)$
$\therefore$ S is LD.

4. Let W ~~be the set~~ of all $(x_1, x_2, x_3, x_4, x_5)$ in R^5 which satisfy
$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$
$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$
$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Find a finite set of vectors which spans W.

Echelon form of a matrix

A matrix 'A' is said to be in echelon form if the number of zeroes preceding the non-zero elt of a row increases row by row and the elts of last row or rows may be all zeroes.

Ex: $\begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

are all echelon matrices.

Note:

$\boxed{1}$ The rank of matrix in echelon form is equal to the no. of non-zero rows of the matrix.

Ex: $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly the matrix A in echelon form

$\therefore$ The no. of non-zero rows in echelon form $= 2$

$\therefore \rho(A) = 2$

Note $\boxed{2}$ Let

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= b_1 \\ a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}\right\} \text{—— (I)}$$

given system of 3 non-homogeneous linear equations in 3 unknowns x, y, z.

Now write the single matrix equation

$$AX = B$$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}_{3\times 1}$

and the matrix $[A|B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$ is called the augmented matrix of the given system of equations.

* working rule for finding the solutions of the equation $AX=B$:-

→ Now the augmented matrix $[A|B]$ Reduce to an echelon form by appling only elementary row operations.

→ This echelon form will enable us to know the ranks of the augmented matrix $[A|B]$ and the coefficient matrix A.

Then the following cases arise:

(i) If $\rho(A) = \rho(A|B) =$ the no. of unknowns.

then the given system (I) is consistent and has unique solution.

(ii) If $\rho(A) = \rho(A|B) <$ no. of unknowns.

then the given system (I) is consistent and has infinite solutions.

(iii) If $\rho(A) \neq \rho(A|B)$ then the given system is inconsistent and has no solution.

Note [13]. Let

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z &= 0 \\ a_{21}x + a_{22}y + a_{23}z &= 0 \\ a_{31}x + a_{32}y + a_{33}z &= 0 \end{aligned}\right\} \text{—— (II)}$$

be the given system of 3 homogeneous linear equations in 3 unknowns x, y, z.

Now write the single matrix equation

$$AX = O$$

where coefficient matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times 1} ; \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$$

Working rule for finding the solutions of the equation $AX=0$:-

→ Reduce the coefficient matrix A to echelon form by applying elementary row operations only.

This echelon form will help us to know the rank of the matrix A.

→ If $\rho(A)$ = no. of unknowns,

then the system (I) possesses a zero solution (trivial solution)

i.e., $x=0, y=0, z=0$

→ If $\rho(A)$ < no. of unknowns.

then there will be a non-zero solution (non-trivial solution)

Problem → Determine whether or not $\alpha=(3,9,-4,-2)$ in $\mathbb{R}^4$ is a linear combination of $\alpha_1=(1,-2,0,3)$, $\alpha_2=(2,3,0,-1)$ and $\alpha_3=(2,-1,2,1)$

Sol:- Let $x, y, z \in \mathbb{R}$.

$\alpha = x\alpha_1 + y\alpha_2 + z\alpha_3$

$\Rightarrow (3,9,-4,-2) = x(1,-2,0,3) + y(2,3,0,-1) + z(2,-1,2,1)$

$\Rightarrow x+2y+2z=3$

$-2x+3y-z=9$

$2z=-4$

$3x-y+z=-2$

Now write the single matrix equation $AX=B$

i.e.,
$$\begin{bmatrix} 1 & 2 & 2 \\ -2 & 3 & -1 \\ 0 & 0 & 2 \\ 3 & -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -4 \\ -2 \end{bmatrix}$$

Augmented matrix
$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ -2 & 3 & -1 & 9 \\ 0 & 0 & 2 & -4 \\ 3 & -1 & 1 & -2 \end{array}\right]$$

$R_2 \to R_2 + 2R_1$
$R_4 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & -7 & -5 & -11 \end{array}\right]$$

$R_4 \to R_4 + R_2$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\begin{bmatrix} 1 & 2 & 2 & 3 \\ 0 & 7 & 3 & 15 \\ 0 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = \rho(A/B) = 3 =$ no. of unknowns x, y, z.

$\therefore$ The given system is consistent and has unique solⁿ.

for solving the unknowns x, y, z..
we write the echelon matrix equation

$$\begin{bmatrix} 1 & 2 & 2 \\ 0 & 7 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 15 \\ -4 \\ 0 \end{bmatrix}$$

$\Rightarrow x + 2y + 2z = 3$

$7y + 3z = 15$

$2z = -4 \Rightarrow \boxed{z = -2}$; $\boxed{x = 1}$ and $\boxed{y = 3}$

$\therefore \alpha = 1\alpha_1 + 3\alpha_2 - 2\alpha_3$.

$\therefore \alpha$ is a linear combination of $\alpha_1, \alpha_2, \alpha_3$.

Note: [1]. If the system of linear equations are consistent then it has ~~unique~~ a solⁿ and the vector α is a linear combination of α_i $(1 \le i \le n)$.

[2]. If the given system of linear equations are not consistent then it has no solution and the vector α is not a linear combination of α_i $(1 \le i \le n)$

→ P.T the set $\{(-1,2,1), (3,0,-1), (-5,4,3)\} \subseteq \mathbb{R}^3$ is LD. 38

Sol^n Let $a, b, c \in \mathbb{R}$ then

$$a(-1,2,1)+b(3,0,-1)+c(-5,4,3) = (0,0,0).$$

$$\Rightarrow -a+3b-5c=0 \quad —(1)$$
$$2a+4c=0 \quad —(2)$$
$$a-b+3c=0 \quad —(3).$$

Solving the above equations, we get

(1)+(3) ≡ $2b-2c=0 \Rightarrow \boxed{b=c}$

(2) ≡ $\boxed{a=-2c}$

(3) ≡ $-2c-c+3c=0$
$\Rightarrow 0=0$

∴ ∃ non-zero values for a, b, c to satisfy the equations (1), (2), (3).

∴ The given set is LD.

→ Determine whether or not the vectors $(1,-2,1), (2,1,-1)$, $(7,-4,1)$ are LD.

Sol^n If $a, b, c \in F$, then

$$a(1,-2,1)+b(2,1,-1)+c(7,-4,1) = (0,0,0).$$

$$\Rightarrow \left.\begin{array}{l} a+2b+7c=0 \\ -2a+b-4c=0 \\ a-b+c=0 \end{array}\right\} —(1)$$

Coefficient matrix $A = \begin{bmatrix} 1 & 2 & 7 \\ -2 & 1 & -4 \\ 1 & -1 & 1 \end{bmatrix}$

Now $|A| = 1(-3)-2(2)+7(1)$
$= -3-4+7=0$

∴ $\rho(A) <$ no. of unknowns a, b, c.

∴ The system of equations possess a non-zero solution.

∴ The given vectors are LD.

Note: [1]. Consider the system of three linear equations in three unknown variables.

$$\left.\begin{array}{l} a_{11}x+a_{12}y+a_{13}z=0 \\ a_{21}x+a_{22}y+a_{23}z=0 \\ a_{31}x+a_{32}y+a_{33}z=0 \end{array}\right\} —(1)$$

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

→ If $|A| \neq 0$,
then the system (i) possesses a trivial solution (zero soln)
i.e, $x=0, y=0, z=0$.

→ If $|A| = 0$, the system (i) possesses a non trivial solution (non-zero soln).

→ Determine whether $(2,-3,7)$, $(0,0,0)$, $(3,-1,-4)$ are LD.

$\boxed{\text{Method (1)}}$

Soln: Let $a, b, c \in \mathbb{R}$. then

$a(2,-3,7) + b(0,0,0) + c(3,-1,-4) = (0,0,0)$.

$$\Rightarrow \left.\begin{aligned} 2a + 0b + 3c &= 0 \\ -3a + 0b - c &= 0 \\ 7a + 0b - 4c &= 0 \end{aligned}\right\} \text{——} (1)$$

The coefficient matrix $A = \begin{bmatrix} 2 & 0 & 3 \\ -3 & 0 & -1 \\ 7 & 0 & -4 \end{bmatrix}$

and $|A| = 0$

∴ The system of equations possess a non-zero solution.

∴ The given vectors are LD.

$\boxed{\text{Method (2)}}$ form the matrix 'A' whose rows are the given vectors

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 0 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

$\Rightarrow |A| = 0$

∴ The given vectors are LD.

$\boxed{\text{Method (3)}}$

form the matrix 'A' whose rows are the given vectors and reduce to echelon form

$$A = \begin{bmatrix} 2 & -3 & 7 \\ 0 & 0 & 0 \\ 3 & -1 & -4 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$ $\begin{bmatrix} 2 & -3 & 7 \\ 3 & -1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$

$R_2 \to 2R_2 - 3R_1$ $\begin{bmatrix} 2 & -3 & 7 \\ 0 & 7 & -29 \\ 0 & 0 & 0 \end{bmatrix}$

which is an echelon form.

Since the echelon form has a zero row.

$\therefore$ The given vectors are LD.

→ In $V_3(\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers, examine each of the following sets of vectors for linear dependent.

(i) $\{(2,1,2), (8,4,8)\}$ (ii) $\{(1,2,0), (0,3,1), (-1,0,1)\}$

(iii) $\{(2,3,5), (4,9,25)\}$.

→ P.T the set $\{(1,2,1), (3,1,5), (2,-4,7)\} \subseteq \mathbb{R}^3$ is LI.

→ Examine the vectors $(1,1,2,4), (2,-1,-5,2), (1,-1,-4,0), (2,1,1,6)$ are LI in $\mathbb{R}^4$.

Soln: Now form the matrix 'A' whose rows are given vectors and reduce to echelon form.

$$A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 2 & -1 & -5 & 2 \\ 1 & -1 & -4 & 0 \\ 2 & 1 & 1 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 2R_1$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -9 & -6 \\ 0 & -2 & -6 & -4 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_2 \to -\frac{1}{3}R_2$
$R_3 \to \frac{1}{2}R_3$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -1 & -3 & -2 \\ 0 & -1 & -3 & -2 \end{bmatrix}$

$R_3 \to R_3 + R_2$
$R_4 \to R_4 + R_2$ $\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Clearly which is in echelon form.

Since this echelon form has two zero rows.

$\therefore$ The given vectors are LD

→ Determine whether $(1,2,-3), (1,-3,2), (2,-1,5)$ are L.D.

Sol^n: Now form the matrix A whose rows are given vectors and reduce to echelon form.

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 1 & -3 & 2 \\ 2 & -1 & 5 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & -5 & 11 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 1 & 2 & -3 \\ 0 & -5 & 5 \\ 0 & 0 & 6 \end{bmatrix}$

Clearly which is in echelon form.

Since the echelon form has no zero rows.

∴ The given vectors are L.I.

M99 → Let V be the vector space of functions from $\mathbb{R} \to \mathbb{R}$.

Show that $f, g, h \in V$ are L.I.

where $f(t) = e^{2t}$, $g(t) = t^2$; $h(t) = t$.

Sol^n: Let $a, b, c \in \mathbb{R}$ then $af + bg + ch = 0$

Now for every ~~value of t,~~

we have $a f(t) + b g(t) + c h(t) = 0$

$\Rightarrow a e^{2t} + bt^2 + ct = 0$

if $t = 0$, then $a e^0 + b(0) + c(0) = 0$

$\Rightarrow \boxed{a = 0}$ ⟵ (1)

if $t = 1$ then $a e^{2(1)} + b(1)^2 + c(1) = 0$

$\Rightarrow ae^2 + b + c = 0$ ⟵ (2)

if $t = 2$ then $a e^{2(2)} + b(2)^2 + c(2) = 0$

$\Rightarrow a e^4 + 4b + 2c = 0$ ⟵ (3)

(2) ≡ $b + c = 0$ ⟵ (4)

(3) ≡ $4b + 2c = 0$ ⟵ (5) $(\because a = 0)$

$2 \times (4) - (5) \equiv -2b = 0$

$\Rightarrow \boxed{b = 0}$

(4) ≡ $\boxed{c = 0}$

∴ f, g, h are L.I.

$\Rightarrow$ S.T the functions $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = t$ are L.I.

Soln: Let $a, b, c \in \mathbb{R}$

then $af + bg + ch = 0$

for every value of t

we have $af(t) + bg(t) + ch(t) = 0$

$\Rightarrow a\sin t + b\cos t + ct = 0$ —— (I)

if $t = 0$ then $a(0) + b(1) + c(0) = 0$

$\Rightarrow \boxed{b = 0}$ —— (1)

if $t = \pi/2$ then $a(1) + b(0) + c(\pi/2) = 0$

$\Rightarrow a + c(\pi/2) = 0$ —— (2)

if $t = \pi$ then $a(0) + b(-1) + c\pi = 0$

$\Rightarrow \boxed{-b + c\pi = 0}$ —— (3)

from (1) & (3) $0 + c\pi = 0$

$\Rightarrow \boxed{c = 0}$ —— (4)

from (2) & (4) $a + 0 = 0$

$\Rightarrow \boxed{a = 0}$

$\therefore f(t), g(t), h(t)$ are L.I.

Second method:

diff (I) three times w.r.t t

—— (1)

—— (2)

—— (3)

2005 $\Rightarrow$ Find the values of K for which the vectors $(1,1,1,1)$, $(1,3,-2,k)$, $(2, 2k-2, -k-2, 3k-1)$ and $(3, k+2, -3, 2k+1)$ are L.I in $\mathbb{R}^4$.

Soln Form the matrix A whose rows are given vectors.

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{bmatrix}$$

Since the given vectors are LI

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & k \\ 2 & 2k-2 & -k-2 & 3k-1 \\ 3 & k+2 & -3 & 2k+1 \end{vmatrix} \neq 0$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 3R_1$

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & k-1 \\ 0 & 2k-4 & -k-4 & 3k-3 \\ 0 & k-1 & -6 & 2k-2 \end{vmatrix} \neq 0$$

$$\Rightarrow 1\begin{vmatrix} 2 & -3 & k-1 \\ 2k-4 & -k-4 & 3k-3 \\ k-1 & -6 & 2k-2 \end{vmatrix} \neq 0$$

Proceeding in this way.

→ If α_1 & α_2 are vectors of $V(F)$ and $a, b \in F$.
S.T the set $\{\alpha_1, \alpha_2, a\alpha_1 + b\alpha_2\}$ is LD.

Sol^n Let $S = \{\alpha_1, \alpha_2, a\alpha_1 + b\alpha_2\} \subseteq V(F)$.
Since one of the vector of S is a l.c. of the remaining vectors.
i.e., $a\alpha_1 + b\alpha_2 = a\alpha_1 + b\alpha_2$
$\therefore$ S is LD.

→ Let $\alpha_1, \alpha_2, \alpha_3$ be vectors of $V(F)$, $a, b \in F$.
S.T. the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is LD if the set $\{\alpha_1 + a\alpha_2 + b\alpha_3, \alpha_2, \alpha_3\}$ is LD.

Sol^n Since the set $\{\alpha_1 + a\alpha_2 + b\alpha_3, \alpha_2, \alpha_3\} \subseteq V$ is LD.
$\therefore \exists$ atleast one non-zero scalar $x, y, z \in F$ s.t
$x(\alpha_1 + a\alpha_2 + b\alpha_3) + y(\alpha_2) + z(\alpha_3) = 0$
$\Rightarrow x\alpha_1 + (ax + y)\alpha_2 + (bx + z)\alpha_3 = 0$
If $x \neq 0$ then the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is LD.
If $x = 0$ then atleast one of y & z is not zero
$\therefore$ Atleast one of $ax + y$ & $bx + z$ is not zero.
$\therefore$ the set $\{\alpha_1, \alpha_2, \alpha_3\}$ is LD.

⇒ If α, β, γ are LI vectors of $V(F)$. Where F is field of Complex numbers then $\alpha + \beta, \beta + \gamma, \gamma + \alpha$ are also LI.

Sol^n Let $a, b, c \in F$ then
$a(\alpha + \beta) + b(\beta + \gamma) + c(\gamma + \alpha) = 0$
$\Rightarrow (a + c)\alpha + (a + b)\beta + (b + c)\gamma = 0$
Since α, β, γ are LI
$\therefore a + c = 0$ — ①
$a + b = 0$ — ②
$b + c = 0$ — ③
① − ③ ≡ $a - b = 0$ — ④
② + ④ ≡ $2a = 0 \Rightarrow \boxed{a = 0}$
④ ≡ $\boxed{b = 0}$ and ③ ≡ $\boxed{c = 0}$
$\therefore \alpha + \beta, \beta + \gamma, \gamma + \alpha$ are LI.

→ Let $C(C)$ be a vector space. then show that $\{1, i\} \subseteq C(C)$ is LD.

Sol: Let $S = \{1, i\} \subseteq C(C)$

Since one of the vector of S is scalar multiple of other.

i.e, $i = i(1)$

$\therefore S$ is LD.

→ Let $C(\mathbb{R})$ be a vector space then show that $\{1, i\} \subseteq C(\mathbb{R})$ is LI.

Sol: Let $a, b \in \mathbb{R}$ then $a(1) + b(i) = 0 + 0(i)$

$\Rightarrow a = 0, b = 0$

$\therefore \{1, i\}$ is LI.

→ S.T the set $\{(1+i, 2i), (1, 1+i)\} \subseteq C^2(C)$ is LD. over the field of complex numbers.

Sol: Let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq C^2(C)$

— Since one of the vector of S is a scalar multiple of other.

i.e, $(1+i, 2i) = (1+i)(1, 1+i)$

$\therefore S$ is LD.

→ S.T $\{(1+i, 2i), (1, 1+i)\} \subseteq C^2(\mathbb{R})$ is LI. over the field of real numbers.

Sol: Let $S = \{(1+i, 2i), (1, 1+i)\} \subseteq C^2(\mathbb{R})$

Let $a, b \in \mathbb{R}$ then

$a(1+i, 2i) + b(1, 1+i) = (0, 0)$

$\Rightarrow (a+ia, 2ia) + (b, b+ib) = (0, 0)$

$\Rightarrow (a+ia+b, 2ia+b+ib) = (0, 0)$

$\Rightarrow a(1+i) + b = 0$ —(1)

$b(1+i) + 2ia = 0$ —(2)

(1) $\equiv (a+b) + ia = 0 + i0$

$\Rightarrow a+b = 0$ & $\boxed{a = 0}$

$\Rightarrow \boxed{b = 0}$

$\therefore S$ is LI.

→ In the vector space $F[x]$ of all polynomials over the field F, the infinite set $S = \{1, x, x^2, \ldots\}$ is L.I.

Sol^n Let $S' = \{x^{m_1}, x^{m_2}, \ldots \ldots x^{m_n}\}$ be any finite subset of S having n vectors.

Hence $m_1, m_2, \ldots m_n$ are non-negative integers

Let $a_1, a_2, \ldots a_n \in F$ s.t

$$a_1 x^{m_1} + a_2 x^{m_2} + \ldots + a_n x^{m_n} = 0x^{m_1} + 0x^{m_2} + \ldots + 0x^{m_n}$$

$$\Rightarrow a_1 = a_2 = \ldots \ldots = a_n = 0$$

$\therefore$ Every finite subset of S is L.I.

$\therefore$ S is L.I.

→ Let $S = \{(1,0), (0,1)\} \subseteq \mathbb{R}^2(\mathbb{R})$. S.T $(3,5) \in L(S)$.

Sol^n $(3,5) = 3(1,0) + 5(0,1)$

$\Rightarrow (3,5) \in L(S)$.

→ Let $S = \{(1,0,0), (0,1,0)\} \subseteq \mathbb{R}^3(\mathbb{R})$. find $L(S)$.
Do $(3,2,0)$, and $(2,5,1)$ belong to $L(S)$.

Sol^n $L(S) = \{\alpha(1,0,0) + \beta(0,1,0) / \alpha, \beta \in \mathbb{R}\} \subseteq \mathbb{R}^3$

$= \{(\alpha, \beta, 0) / \alpha, \beta \in \mathbb{R}\}$

$\therefore (3,2,0) \in L(S)$.

but $(2,5,1) \notin L(S)$. $(\because 1 \neq 0)$

→ Let $S = \{(2,3), (1,4)\} \subseteq \mathbb{R}^2(\mathbb{R})$. S.T $(4,1) \in L(S)$.

Sol^n $(4,1) = \alpha(2,3) + \beta(1,4)$; $\alpha, \beta \in \mathbb{R}$.

$\Rightarrow 2\alpha + \beta = 4$

$3\alpha + 4\beta = 1$

$\Rightarrow \alpha = 3, \beta = -2$.

$\therefore (4,1) = 3(2,3) - 2(1,4)$

$\therefore (4,1) \in L(S)$

→ Is the vector $(2,-5,3)$ in the subspace of $\mathbb{R}^3$ spanned by the vectors $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$?

Sol^n: Let $\alpha = (2,-5,3)$, $\alpha_1 = (1,-3,2)$; $\alpha_2 = (2,-4,-1)$
$\alpha_3 = (1,-5,7)$

$\therefore$ Let $S = \{\alpha_1, \alpha_2, \alpha_3\} \subseteq \mathbb{R}^3(\mathbb{R})$

Let $\alpha = a\alpha_1 + b\alpha_2 + c\alpha_3$; $a, b, c \in \mathbb{R}$

then $(2,-5,3) = a(1,-3,2) + b(2,-4,-1) + c(1,-5,7)$

$\Rightarrow a + 2b + c = 2$ —(1)
$-3a - 4b - 5c = -5$ —(2)
$2a - b + 7c = 3$ —(3)

$3 \times (1) + (2) \Rightarrow 2b - 2c = 1 \Rightarrow b - c = 1/2$ —(4)
$2 \times (1) - (3) \Rightarrow 5b - 5c = 1 \Rightarrow b - c = 1/5$ —(5)

$\therefore$ The equations (4) & (5) are inconsistent.
$\therefore$ α cannot be expressed as l.c. of S.
$\therefore$ α is not in the subspace of $\mathbb{R}^3$ spanned by S.

→ In the vector space $\mathbb{R}^3$.
let $\alpha = (1,2,1)$, $\beta = (3,1,5)$, $\gamma = (3,-4,7)$.
S.T. the subspaces spanned by $S = \{\alpha, \beta\}$ and $T = \{\alpha, \beta, \gamma\}$ are the same.

Sol^n: Let $S = \{\alpha, \beta\} \subseteq V_3(\mathbb{R})$
$T = \{\alpha, \beta, \gamma\} \subseteq V_3(\mathbb{R})$.
and $L(S)$ & $L(T)$ be two subspaces spanned by S & T.

we have to S.T $L(S) = L(T)$.

since $S \subseteq T \Rightarrow L(S) \subseteq L(T)$ —(1)

Let $x \in L(T)$ then

$x = a\alpha + b\beta + c\gamma$; $a, b, c \in \mathbb{R}$.

Let $\gamma = a_1\alpha + a_2\beta$; $a_1, a_2 \in \mathbb{R}$

$\Rightarrow (3,-4,7) = a_1(1,2,1) + a_2(3,1,5)$

$\Rightarrow a_1 + 3a_2 = 3$ — (i)

$2a_1 + a_2 = -4$ — (ii)

$a_1 + 5a_2 = 7$ — (iii)

(i) − (iii) $\equiv -2a_2 = -4$

$\boxed{a_2 = 2}$

and $\boxed{a_1 = -3}$

$\therefore$ (3) $\equiv \gamma = -3\alpha + 2\beta$.

$\therefore$ (2) $\equiv x = a\alpha + b\beta + c(-3\alpha + 2\beta)$

$= (a - 3c)\alpha + (b + 2c)\beta$

$=$ L.C. of α & β.

$\therefore x \in L(S)$.

$\therefore L(T) \subseteq L(S)$. — (4)

$\therefore$ from (1) & (4)

we have $L(S) = L(T)$

$\longrightarrow$ Is the vector $(3, -4, 6)$ in the subspace of R^3 spanned by the vectors $(1, 2, -1)$, $(2, 2, 1)$ and $(1, -2, 3)$?

$\longrightarrow$ Let $\alpha_1 = (1, 1, -2, 1)$, $\alpha_2 = (3, 0, 4, -1)$, $\alpha_3 = (-1, 2, 5, 2)$.
Show that $(4, -5, 9, -7)$ is spanned by $\alpha_1, \alpha_2, \alpha_3$.

$\longrightarrow$ Is the vector $(3, -1, 0, -1)$ in the subspace of R^4 spanned by the vectors $\alpha_1 = (2, -1, 3, 2)$, $\alpha_2 = (-1, 1, 1, -3)$ and $\alpha_3 = (1, 1, 9, -5)$?

$\longrightarrow$ Let $V = R^3(R)$ and $S = \{\alpha_1 = (1,1,0), \alpha_2 = (0,-1,1), \alpha_3 = (1,0,1)\}$
Prove that $(a, b, c) \in L(S)$ iff $a = b + c$.

Sol: By definition of $L(S)$, $(a, b, c$ —

$(a, b, c) \in L(S)$

$\Leftrightarrow (a, b, c) = \alpha(1, 1, 0) + \beta(0, -1, 1) + \gamma(1, 0, 1)$; $\alpha, \beta, \gamma \in R$

$\Leftrightarrow (a, b, c) = (\alpha + \gamma, \alpha - \beta, \beta + \gamma)$

$\Leftrightarrow a = \alpha + \gamma$, $b = \alpha - \beta$, $c = \beta + \gamma$.

$\Leftrightarrow a = b + c$

→ If v_1, v_2, v_3 are three vectors in a vector space $V(F)$ such that $v_1+v_2+v_3=0$, then show that $\{v_1, v_2\}$ spans the same subspace as $\{v_2, v_3\}$.

Solⁿ: Let $S=\{v_1, v_2\}$ and $T=\{v_2, v_3\}$.

We shall prove that $L(S)=L(T)$

Let $x \in L(S)$.

Then $x = \alpha v_1 + \beta v_2$; $\alpha, \beta \in F$

$\Rightarrow x = \alpha(-v_2 - v_3) + \beta v_2$

$\Rightarrow x = (\beta - \alpha) v_2 - \alpha v_3 \in L(T)$

$\therefore L(S) \subseteq L(T)$.

Conversely, let $y \in L(T)$

Then $y = a v_2 + b v_3$; $a, b \in F$

$\Rightarrow y = a v_2 + b(-v_1 - v_2)$

$\Rightarrow y = -b v_1 + (a-b) v_2 \in L(S)$

$\therefore L(T) \subseteq L(S)$.

Hence $L(S) = L(T)$

→ Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Find a finite set of vectors which spans W.

Basis and Dimension: Set - II   Vector Space - I

Basis: Let $V(F)$ be a vector space and $S=\{\alpha_1,\alpha_2,\dots,\alpha_n\}\subseteq V$

if (i) S is LI

(ii) $L(S)=V$ i.e, V spanned by S.

i.e, each vector in V is a l.c. of finite no. of elts of S.

then S is called basis of $V(F)$.

Ex:- $S=\{e_1,e_2,\dots e_n\}\subseteq V_n(F)$

where $e_1=(1,0,0,\dots 0)$; $e_2=(0,1,0,\dots 0)\dots e_n=(0,0,\dots 1)$

is basis of $V_n(F)$.

Sol^n (i) To prove S is LI.

$S=\{e_1,e_2\dots e_n\}\subseteq V_n(F)$

where $e_1=(1,0,\dots 0)$, $e_2=(0,1,0\dots)\dots e_n=(0,0,\dots 1)$

Let $a_1,a_2,\dots a_n\in F$ then

$a_1e_1+a_2e_2+\dots+a_ne_n=0$

$\Rightarrow a_1(1,0,\dots 0)+a_2(0,1,0,\dots 0)+\dots+a_n(0,0,\dots 0,1)$
$=(0,0,\dots,0)$

$\Rightarrow (a_1,a_2,\dots a_n)=(0,0,0,\dots 0)$

$\Rightarrow a_1=a_2=a_3=\dots=a_n=0$

$\therefore S$ is LI.

(ii) To prove that, $L(S)=V_n(F)$.

we have always $L(S)\subseteq V_n(F)$ —— (1)

Let $\alpha=(a_1,a_2,\dots a_n)=a_1(1,0,0\dots 0)+a_2(0,1,0,\dots 0)$
$+\dots+a_n(0,0\dots 0,1)\in L(S)$

$\therefore \alpha\in L(S)$

$\Rightarrow V_n(F)\subset L(S)$ —— (2)

$\therefore$ from (1) & (2) $L(S)=V_n(F)$.

$\therefore S$ is a basis of $V_n(F)$

Note [1]. The set $\{(1,0,0,\dots 0),(0,1,0,\dots 0),\dots\dots(0,0,\dots 0,1)\}\subseteq V_n(F)$ is called the standard basis of $V_n(F)$.

[2]. $\{(1,0),(0,1)\}$ is a basis of $V_2(F)$

[3]. $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $V_3(F)$.

Ex:- $S=\{1,i\}$ is a basis of $C(\mathbb{R})$.

Sol^n: S is L.I

we have always $L(S) \subseteq C(\mathbb{R})$ — ①

Let $\alpha \in C(\mathbb{R})$ then $\alpha = a+bi$; $a, b \in \mathbb{R}$

$= a\cdot 1 + b(i) \in L(S)$

$\therefore \alpha \in L(S)$

$\therefore C(\mathbb{R}) \subseteq L(S)$ — ②

$\therefore$ from (1) & (2) $L(S) = C(\mathbb{R})$

$\therefore$ S is a basis of $C(\mathbb{R})$

Ex:- Let $F_3[x] = \{a_0+a_1x+a_2x^2 / a_0, a_1, a_2 \in F\}$

then $\{1, x, x^2\} \subseteq F_3[x]$ is basis of $F_3[x]$ over F.

Sol^n Let $S=\{1, x, x^2\} \subseteq F[x]$

S is L.I

we ~~have~~ always $L(S) \subseteq F[x]$ — ①

Let $\alpha = a_0+a_1x+a_2x^2 \in F[x]$

then $a_0+a_1x+a_2x^2 = a_0(1)+a_1(x^2)+a_2(x^3) \in L(S)$

$\therefore \alpha \in L(S)$

$\therefore F[x] \subseteq L(S)$ — ②

$\therefore$ from (1) & (2) $L(S) = F[x]$

$\therefore$ S is a basis of $F[x]$.

→ S.T the set $\{(1,0,0),(0,1,0),(1,1,0),(1,2,3)\} \subseteq V_3(\mathbb{R})$ is not a basis of $V_3(\mathbb{R})$

Sol^n Let $S=\{(1,0,0),(0,1,0),(1,1,0),(1,2,3)\} \subseteq V_3(\mathbb{R})$

(i) To check whether the set S is LI or not:

Let $a, b, c, d \in \mathbb{R}$ then

$a(1,0,0)+b(1,1,0)+c(1,2,3)+d(0,1,0) = (0,0,0)$

$\Rightarrow (a+b+c,\ b+2c+d,\ 3c) = (0,0,0)$

$\Rightarrow a+b+c=0$ —(1) $\quad 3c=0$

$b+2c+d=0$ —(2) $\quad \Rightarrow \boxed{c=0}$

① ≡ $a+b=0 \Rightarrow \boxed{a=-b}$

② ≡ $b+d=0 \Rightarrow \boxed{b=-d}$

If $d=K\neq 0$, then $b=-K$ and $a=K$

∴ ∃ non-zero values for a, b, d to satisfy the equations ①, ②

∴ The given set of vectors are LD.

∴ S is not a basis set of $V_3(\mathbb{R})$

Note: Any subset of $V_n(F)$, (i.e., $S\subseteq V_n(F)$) having more than n elts will be LD and it cannot be a basis set of $V_n(F)$.

Defn: Finite Dimensional vector space (FDVS)

→ The vector space $V(F)$ is said to finite dimensional vector space or finitely generated if there exists a finite subset S of V s.t $V=L(S)$.

Note: If there exists no finite subset which spans V then V is called an infinite dimensional vectorspace

Ex: Let $S=\{(1,0),(0,1)\}\subseteq V_2(F)$ then $V_2(F)$ is FDVS.

Sol: Let $(a,b)\in V_2(F)$; $a,b\in F$

then $(a,b) = x(1,0)+y(0,1)$; $x,y\in F$

$\Rightarrow (a,b) = (x,0)+(0,y)$
$= (x,y)$

$\Rightarrow \boxed{x=a}$; $\boxed{y=b}$

∴ $(a,b) = a(1,0)+b(0,1) \in L(S)$

∴ $(a,b) \in L(S)$

∴ $V_2(F)\subseteq L(S)$ — ①

W.K.T $L(S)\subseteq V_2(F)$ — ②

∴ from ① & ② we have $V_2(F) = L(S)$.

∴ $V_2(F)$ is a FDVS.

→ Similarly $V_3(\mathbb{R}) = \{(a,b,c)/ a,b,c\in\mathbb{R}\}$ is a FDVS.

Since $V_3(\mathbb{R}) = L(S)$ where $S=\{(1,0,0),(0,1,0),(0,0,1)\}\subseteq V_3(\mathbb{R})$

$V_n(\mathbb{R}) = \{(a_1, a_2, \ldots, a_n) / a_1, a_2, \ldots a_n \in \mathbb{R}\}$

is a FDVS.

Since $V_n(\mathbb{R}) = L(S)$

where $S = \{(1,0,0,\ldots 0), (0,1,0,\ldots 0), \ldots (0,0,\ldots,0,1)\} \subseteq V_n(\mathbb{R})$

Note: A vector space may have more than one basis.

Ex (1) $S = \{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2(\mathbb{R})$

(2) $T = \{(1,1), (1,0)\}$ is also a basis of $\mathbb{R}^2(\mathbb{R})$

Sol: Let $a, b \in \mathbb{R}$ then

$$a(1,1) + b(1,0) = (0,0)$$
$$\Rightarrow (a+b, a) = (0,0)$$
$$\Rightarrow a+b=0, \boxed{a=0}$$
$$\Rightarrow \boxed{b=0}$$

$\therefore$ T is LI.

W.K.T $L(T) \subseteq \mathbb{R}^2(\mathbb{R})$ — (1)

Let $(a,b) \in \mathbb{R}^2(\mathbb{R})$ then

$$(a,b) = b(1,1) + (a-b)(1,0)$$

$\therefore \mathbb{R}^2(\mathbb{R}) \subseteq L(T)$ — (2)

$\therefore$ from (1) & (2) $\mathbb{R}^2(\mathbb{R}) = L(T)$.

$\therefore$ T is a basis of $\mathbb{R}^2(\mathbb{R})$.

→ Show that the set $S = \{1, x, x^2, \ldots x^n\}$ of $n+1$ polynomials is a basis for the vector space $F_n[x]$ of all polynomials of degree n over the field F.

Sol: Given that $S = \{1, x, x^2, \ldots x^n\} \subseteq F_n[x]$

(i) To prove S is LI

Let $a_0, a_1, a_2, \ldots a_n \in F$ then

$$a_0(1) + a_1(x) + a_2 x^2 + \cdots + a_n(x^n) = 0 \text{ (zero polynomial)}$$
$$\Rightarrow a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = 0 + 0x + 0x^2 + \cdots + 0x^n$$
$$\Rightarrow a_0 = a_1 = a_2 = \cdots = a_n = 0$$

$\therefore$ S is LI

(ii) To prove $L(S) = F_n[x]$

W.K.T $L(S) \subseteq F_n[x]$

Let $f(x)$ be any polynomial of degree n over F.

i.e, $f(x) \in F_n[x]$.

then $f(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$; where $b_0, b_1, \ldots b_n \in F$

l.c.s. of elts of S
$\in L(S)$
$\therefore f(x) \in L(S)$
$\therefore$ If $f(x) \in F_n[x]$ then $f(x) \in L(S)$
$\therefore F_n[x] \subseteq L(S)$ —(2)
from (1) & (2) $L(S) = F_n[x]$
$\therefore$ S is a basis of $F_n[x]$.

Note: The above basis 'S' is the standard basis of the vector space of all polynomials of degree n over F.

Infinite dimensional vector space :-

Defn: The vector space $V(F)$ is said to be infinite dimensional vector space or infinitely generated if there exists an infinite subset S of V s.t $L(S) = V$.

Ex: Show that the set $S = \{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis of the vector space $F[x]$ of all polynomials over the field F

Sol^n Given that $S = \{1, x, x^2, \ldots x^n\} \subseteq F[x]$.

(i) To prove S is LI.

$S' = \{x^{m_1}, x^{m_2}, \ldots x^{m_n}\}$ be a finite subset of S having n vectors.

Here $m_1, m_2, \ldots m_n$ are non-negative integers.

Let $a_1, a_2, \ldots a_n \in F$ then $a_1 x^{m_1} + a_2 x^{m_2} + \ldots + a_n x^{m_n} = 0$ (zero poly)

$\Rightarrow a_1 x^{m_1} + a_2 x^{m_2} + \ldots + a_n x^{m_n} = 0x^{m_1} + 0x^{m_2} + \ldots + 0x^{m_n}$

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$

$\therefore S'$ is LI

$\therefore$ Every finite subset of S is LI.

$\therefore S$ is LI.

(ii) To prove $L(S) = F[x]$.

W.K.T $L(S) \subseteq F[x]$

Let $f(x) \in F[x]$ be any polynomial of degree m in $F[x]$

i.e. $f(x) = b_0 + b_1 x + b_2 x^2 + \ldots + b_m x^m$

$= b_0(1) + b_1(x) + b_2x^2 + \cdots + b_mx^m + 0x^{m+1} + 0x^{m+2} + \cdots$

$=$ L.C. of elts of S

$\in L(S)$

$\therefore f(x) \in L(S)$

$\therefore$ If $f(x) \in F[x]$ then $f(x) \in L(S)$

$\therefore F[x] \subseteq L(S)$ ——— (2)

$\therefore$ from (1) & (2) we have $L(S) = F[x]$.

$\therefore S$ is a basis of $F[x]$.

Note: (1). The vector space $F[x]$ is an infinite dimensional vectorspace. Because there exists no finite subsets of $F[x]$ which spans $F[x]$.

(2). The vector space $F[x]$ has no finite basis.

Existence of basis of a finite dimensional vectorspace

Theorem: Every finite dimensional vector space $V(F)$ has a basis (or)

If $S = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ spans $V(F)$. i.e, $L(S) = V$ then there exists a subset of S which forms a basis of V.

Proof: Let $V(F)$ be a finite dimensional vectorspace. s.t $L(S) = V$.

then $\exists$ a finite subset S of V s.t $L(S) = V$.

i.e, Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_m\} \subseteq V$ s.t $L(S) = V$.

If S is LI then S itself is a basis of V.

If S is LD then there exists a vector $\alpha_i \in S$ is a linear combination of its preceding vectors $\alpha_1, \alpha_2, \ldots \alpha_{i-1}$

i.e, $\alpha_i = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1}$ ——— (I)

where $a_1, a_2, \ldots a_{i-1} \in F$

Now if we omit this vector α_i from the set 'S' then the remaining set 'S'' having $m-1$ vectors $\alpha_1, \alpha_2, \ldots \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_m$

i.e, $S' = \{\alpha_1, \alpha_2, \ldots \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_m\} \subseteq S$

Clearly $S' \subset S \Rightarrow L(S') \subset L(S)$

$\Rightarrow L(S') \subset V$ —— (1) $(\because L(S) \subset V)$

Let $\alpha \in V$ then α is l.c. of elts of S.

$\therefore \alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_i\alpha_i + b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

where $b_1, b_2, \ldots b_{i-1}, b_i, b_{i+1}, \ldots \in F$

(I) $\Rightarrow \alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{i-1}\alpha_{i-1} + b_i(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1})$

$+ b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

$= (b_1 + b_i a_1)\alpha_1 + (b_2 + b_i a_2)\alpha_2 + \cdots + (b_{i-1} + b_i a_{i-1})\alpha_{i-1}$

$+ b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

$=$ l.c. of $\alpha_1, \alpha_2, \ldots \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_m$

$=$ l.c. of elts of the set S'

$\in L(S')$

$\therefore \alpha \in L(S')$

$\therefore V \subseteq L(S')$ —— (2)

$\therefore$ from (1) & (2)

$V = L(S')$.

If S' is LI then S' is a basis of $V(F)$.

If S' is LD then proceeding as above we get new set S'' of $m-2$ vectors which generates V. i.e, $L(S'') = V$.

Continuing in this way, after finite no. of steps, obtain a LI subset of S which generates V and therefore it is a basis of V.

At the most repeating the procedure we left with a subset having a single non-zero vector which generates V and we know that a set containing a single non-zero vector is LI.

$\therefore$ It forms a basis of V

Theorem: If $V(F)$ is a finite dimensional vector space, then any two bases of V have same number of elements.

Proof: Let $V(F)$ be a finite dimensional vector space then it has a basis.

Let $S_1 = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ and $S_2 = \{\beta_1, \beta_2 \ldots \beta_n\}$ be two bases of V.

Now we shall prove that $m=n$.

If possible let $m \neq n$ then $m>n$ or $m<n$

Suppose $m>n$:

Since $\alpha_i \in V$ and S_2 is a basis of V, $\exists$ $a_{ji} \in F$ s.t

$$\alpha_i = a_{1i}\beta_1 + a_{2i}\beta_2 + \cdots + a_{ni}\beta_n ;\ i=1,2,\ldots m \quad \text{—(1)}$$

Now consider the relation

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m = 0,\ x_i \in F \quad \text{—(2)}$$

from (1) & (2) we have

$$x_1(a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 + \cdots + a_{n1}\beta_n) + x_2(a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 + \cdots a_{n2}\beta_n)$$
$$+ \cdots + x_m(a_{1m}\beta_1 + a_{2m}\beta_2 + \cdots + a_{nm}\beta_n) = 0$$

$$\Rightarrow (x_1a_{11} + x_2a_{12} + \cdots + x_ma_{1m})\beta_1 + (x_1a_{21} + x_2a_{22} + \cdots + x_ma_{2m})\beta_2$$
$$+ \cdots + (x_1a_{n1} + x_2a_{n2} + \cdots + x_ma_{nm})\beta_n = 0 \quad \text{—(3)}$$

Since $\beta_1, \beta_2, \ldots \beta_n$ are L.I.

from (3) we have

$$\left.\begin{aligned} x_1a_{11} + x_2a_{12} + \cdots + x_ma_{1m} &= 0 \\ x_1a_{21} + x_2a_{22} + \cdots + x_ma_{2m} &= 0 \\ &\ \ \vdots \\ x_1a_{n1} + x_2a_{n2} + \cdots + x_ma_{nm} &= 0 \end{aligned}\right\} \text{—(4)}$$

$$\Rightarrow \left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\ &\ \ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0 \end{aligned}\right\} \text{—(5)}$$

$\therefore$ This is a system of n homogeneous linear eqns in m unknown variables.

As $m>n$ i.e, $n<m$

i.e, no. of eqns are less than no. of unknowns.

$\therefore$ The above system (5) of eqns have a non-zero solution.

i.e, there exist $x_1, x_2, \ldots x_m$ in F not all zero to satisfy the eqn (2).

$\therefore \alpha_1, \alpha_2, \ldots \alpha_m$ are LD

which contradicts that S_1 is a basis of $V(F)$.

$\therefore m \not> n$.

Similarly $m \not< n$.

$\therefore m = n$

i.e, Any two bases of a FDVS $V(F)$ have the same no. of elts.

Dimension of a vector space:

Defn: The no. of elts in any basis of a finite dimensional vector space $V(F)$ is called the dimension of the vector space $V(F)$ and is denoted by $\dim V$ or $\dim_F V$.

Note: [1]. If a vectorspace $V(F)$ has a finite basis having n vectors then $\dim V = n$

[2]. If $\dim V = n$ then V has a basis containing n vectors say $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$

it means the vectors $\alpha_1, \alpha_2, \ldots \alpha_n$ are L.I. and each vector $\alpha \in V$ is expressible as

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $a_1, a_2, \ldots a_n \in F$

Ex: 1) $\dim \mathbb{R}^2 = 2$.

Since $\{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2$

2) $\dim \mathbb{R}^3 = 3$

Since $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $\mathbb{R}^3$

3) $\dim \mathbb{R}^n = n$

Since $\{(1,0,0\ldots 0),(0,1,0\ldots 0);(\ldots (0,0,\ldots 1)\}$ is a basis of $\mathbb{R}^n$.

4) $\dim_{\mathbb{R}} \mathbb{C} = 2$

Since $\{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$.

5). If F is any field then $\dim_F F = 1$

Since $\{1\}$, a set consisting of the unity elt of F is a basis of F over F.

Similarly $\dim_{\mathbb{R}} \mathbb{R} = 1$; $\dim_{\mathbb{C}} \mathbb{C} = 1$.

Note: Every non-zero elt of F will form a basis of F.

→ A finite dimensional vector space V(F) has dimension n iff n is the maximum no. of linearly independent vectors in any subset of V.

Proof: N.C: Let dim V = n and let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of V. Then $\alpha_1, \alpha_2, \dots, \alpha_n$ are L.I.

Let $T = \{\beta_1, \beta_2, \dots, \beta_m\}$ be any subset of V s.t. $m > n$.

If we prove that T is LD set then n is maximum no. of LI vectors in any subset of V.

Since $\beta_i \in V$ and S is a basis of V,

$\exists\ a_{ji} \in F$ s.t.

$$\beta_i = a_{1i}\alpha_1 + a_{2i}\alpha_2 + \dots + a_{ni}\alpha_n \; ; \; i = 1, 2, \dots m. \quad \text{--- (1)}$$

Consider the relation

$$x_1\beta_1 + x_2\beta_2 + x_3\beta_3 + \dots + x_m\beta_m = 0 \; ; \; x_i \in F \quad \text{--- (2)}$$

from (1) & (2) we have

$$x_1(a_{11}\alpha_1 + a_{21}\alpha_2 + a_{31}\alpha_3 + \dots + a_{n1}\alpha_n) + x_2(a_{12}\alpha_1 + a_{22}\alpha_2 + \dots + a_{n2}\alpha_n)$$
$$+ \dots \dots + x_m(a_{1m}\alpha_1 + a_{2m}\alpha_2 + \dots + a_{nm}\alpha_n) = 0$$

$$\Rightarrow (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1m}x_m)\alpha_1 + (a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m)\alpha_2$$
$$+ \dots \dots + (a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nm}x_m)\alpha_n = 0$$

Since $\alpha_1, \alpha_2, \dots, \alpha_n$ are LI

$$\left.\begin{array}{l} \therefore a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m = 0 \\ \dots\dots\dots\dots \\ \dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nm}x_m = 0 \end{array}\right\} \text{--- (3)}$$

This is a system of n homogeneous linear equations in m unknown variables.

As $m > n$ i.e, $n < m$

i.e, the no. of equations are less than no. of unknowns

∴ The above system (3) of equations have non-zero solution.

i.e., There exist non-zero values of $a_1, a_2, \ldots a_m$ to satisfy the relation (2).

$\therefore \beta_1, \beta_2, \ldots \beta_m$ are LD. $(m > n)$

$\therefore$ n is the maximum no. of LI vectors in any subset of V.

S.C: Let n be the maximum of LI vectors in any subset of V.

Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a LI subset of V.

Now we have to prove that S is a basis of V.

For this we are enough to prove that $V = L(S)$.

Since $S \subset V$

$\therefore L(S) \subset V$ ——— (1)

Let $\alpha \in V$ and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a maximal LI set.

$\therefore T = \{\alpha_1, \alpha_2, \ldots \alpha_n, \alpha\}$ is LD. ——— (2)

$\Rightarrow \exists$ atleast one non-zero scalar $a_1, a_2, \ldots a_n, a \in$

s.t $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n + a\alpha = 0$ ——— (3)

If $a = 0$ then

$a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$ $(\because S$ is LI$)$

$\therefore a = a_1 = a_2 = \ldots = a_n = 0$

which contradicts (3).

$\therefore a \neq 0$

$\therefore$ from (3) we have

$a\alpha = -a_1\alpha_1 - a_2\alpha_2 - \ldots - a_n\alpha_n$

$\Rightarrow \alpha = \left(\frac{-a_1}{a}\right)\alpha_1 + \left(\frac{-a_2}{a}\right)\alpha_2 + \ldots + \left(\frac{-a_n}{a}\right)\alpha_n$

$\Rightarrow \alpha$ is L.C. of elts of S.

$\Rightarrow \alpha \in L(S)$

$\therefore V \subset L(S)$ ——— (4)

from (1) & (4) we have $V = L(S)$

$\therefore S$ is a basis containing n vectors.

$\therefore \dim V = n$

→ **Theorem:**

If $\dim V = n$ then any $n+1$ vectors are LD.

proof: Theorem (I) first part.

→ **Extension theorem:**

Every finite linearly independent subset of a finite dimensional vector space V over F can be extended to form a basis of V

(or).

If V is a finite dimensional vectorspace over F and if $S_1 = \{\alpha_1, \alpha_2, \ldots \alpha_r\}$ is any LI set of vectors in V, Prove that, unless 'S_1' is a basis, we can find the vectors $\alpha_{r+1}, \alpha_{r+2}, \ldots \alpha_n$ in V s.t

$\{\alpha_1, \alpha_2 \ldots \alpha_r, \alpha_{r+1}, \alpha_{r+2}, \ldots$

proof: Let $\dim V = n$, then 'n' is the maximum no. of LI vectors in any subset of V.

Since $S_1 = \{\alpha_1 \alpha_2 \ldots \alpha_r\}$ is any LI set of vectors in V.

If S_1 spans V i.e, $L(S_1) = V$. then it forms a basis of V (here $r = n$)

Let $S_2 = \{\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_r, \alpha_{r+1}, \ldots \alpha_n\}$ be the maximal LI subset of V.

If we P.T $L(S_2) = V$ then S_2 is a basis of V.

Let $\alpha \in V$ then $T = \{\alpha_1, \alpha_2 - \alpha_r, \alpha_{r+1}, \alpha_{r+2} \ldots \alpha_n, \alpha\}$.

which contains $n+1$ (i.e $> n$)

it must be LD.

$\therefore \exists$ atleast one non-zero scalar $a_1, a_2, a_3 \ldots a_n, a \in F$

s.t $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_r\alpha_r + \ldots + a_n\alpha_n + a\alpha = 0$ — (1)

If possible let $a = 0$, then $a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = 0$

$\Rightarrow a_1 = a_2 = \ldots = a_n = 0$ ($\because S_2$ is LI)

$\Rightarrow a_1 = a_2 = a_3 = \ldots = a_n = a = 0$

which is contradiction to T is LD

$\therefore a \neq 0$

(1) $\equiv$ $a\alpha = -(a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n)$

$\Rightarrow \alpha = \left(\frac{-a_1}{a}\right)\alpha_1 + \left(\frac{-a_2}{a}\right)\alpha_2 + \ldots + \left(\frac{-a_n}{a}\right)\alpha_n$

$\in L(S_2)$.

$\therefore V \subset L(S_2)$ — (3)

w.k.T $L(S_2) \subseteq V$ — (4)

from (3) & (4)

$V = L(S_2)$

$\therefore S_2$ is a basis

If $\dim V = n$ and $\{\beta_1, \beta_2, \ldots \beta_m\}$ is L.I subset of V then $m \le n$.

(or)

If $\dim V = n$ then a L.I subset S_1 of V cannot have more than 'n' elements.

proof: Let $\dim V = n$ then 'n' is the maximum no. of L.I vectors in any subset V.

Let $S_1 = \{\beta_1, \beta_2, \ldots \beta_m\}$ be a L.I subset of V.

If it contains more than 'n' elts then S_1 is L.D.

$\therefore$ A L.I subset S_1 of V cannot have more than n elts

Theorem (II)

→ If $\dim V = n$ and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a L.I subset of V then S is a basis of V.

proof: Since S is L.I subset of V, it can be extended to form a basis of V.

Since $\dim V = n$ & S contains n L.I vectors.

$\therefore$ S itself forms a basis of V.

→ Theorem:

If $\dim V = n$ and $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ spans V then S is a basis of V.

proof Since $\dim V = n$

$\therefore$ Any basis of V has exactly n elts.

Since S spans V.

i.e., $L(S) = V$.

$\therefore$ there exists any subset of S which forms a basis of V. (By existence of a basis of a FDVS theorem)

Since no basis of V can have fewer than 'n' elts.

$\therefore$ S itself forms a basis of V.

Note: If a vectorspace $V(F)$ is of dimension 'n' then any set of n linearly independent vectors in V forms a basis of V.

(This result in theorem (II))

Theorem: Let $S=\{\alpha_1,\alpha_2,\ldots\alpha_n\}$ be a basis of a finite dimensional vector space $V(F)$ of dimension 'n'. Then every elt α of V can be uniquely expressed as $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ where $a_1, a_2, \ldots a_n \in F$.

Proof: Since $S=\{\alpha_1 \alpha_2 \ldots \alpha_n\}$ is a basis of V.

$\therefore L(S)=V$.

$\therefore$ Any vector $\alpha \in V$ can be expressed as

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \quad \text{--- (1)}$$

To show that (1) is unique representation:

Let us suppose that

$$\alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n \quad \text{--- (2)}$$

where $b_1, b_2, \ldots b_n \in F$.

from (1) & (2) we have $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$

$\Rightarrow (a_1-b_1)\alpha_1 + (a_2-b_2)\alpha_2 + \cdots + (a_n-b_n)\alpha_n = 0$

$\Rightarrow a_1-b_1=0,\ a_2-b_2=0, \ldots a_n-b_n=0$ ($\because S$ is LI)

$\Rightarrow a_1=b_1;\ a_2=b_2;\ \ldots\ a_n=b_n$.

$\therefore$ (1) is a unique expression of V as a l.c. of $\alpha_1, \alpha_2, \ldots \alpha_n$.

Row Reduced Echelon matrix:-

An echelon matrix is called a row reduced echelon matrix or row canonical form iff

(i) The distinguished elts are equal to 1.

and (ii) these elements (distinguished) are the only non-zero elements in their respective coloumns.

Note: The first non-zero elts in the rows of an echelon matrix are called distinguished elts of A.

Ex:- $\begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

are all row reduced echelon matrices

Prove that the non-zero rows of an echelon matrix are LI.

Proof: Let $R_1, R_2, \dots R_{n-1}, R_n$ be the non-zero rows of an echelon matrix A.

If possible let $R_n, R_{n-1} - \dots R_2, R_1$ be the LD.

then one of the rows say R_m is a l.c. of its preceding rows.

i.e, $R_m = a_{m+1} R_{m+1} + a_{m+2} R_{m+2} + \dots + a_n R_n$ — ①

Let k^{th} elt of R_m be its non-zero entry.

Since A is an echelon form,

∴ The k^{th} elt of each $R_{m+1}, R_{m+2}, \dots R_n$ is zero

∴ ① ≡ the k^{th} elt of R_m = k^{th} elt of $[a_{m+1} R_{m+1} + a_{m+2} R_n \dots + a_n R_n]$

$= a_{m+1}(0) + a_{m+2}(0) + \dots + a_n(0)$

$= 0$

∴ k^{th} elt of $R_m = 0$

∴ which contradicts the assumption that k^{th} elt of R_m is non-zero.

∴ $R_1, R_2, \dots R_{n-1}, R_n$ are LI.

problems

① Give examples of two different bases of $V_3(\mathbb{R})$ ~~or~~ R

Sol^n Let $V_3(\mathbb{R}) = \{(a,b,c) / a,b,c \in \mathbb{R}\}$

Let $S_1 = \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq V_3(\mathbb{R})$

and $S_2 = \{(0,1,0), (0,0,1), (2,3,4)\} \subseteq V_3(\mathbb{R})$

Now we show that the sets S_1 & S_2 both form basis for $V_3(\mathbb{R})$.

(I) Let $S_1 = \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq V_3(\mathbb{R})$

(i) To show S_1 is LI.

Let $a_1, a_2, a_3 \in F$ then

$a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (0,0,0)$

$\Rightarrow (a_1,0,0) + (0,a_2,0) + (0,0,a_3) = (0,0,0)$

$\Rightarrow (a_1,a_2,a_3) = (0,0,0)$

$\Rightarrow a_1 = a_2 = a_3 = 0$

$\therefore S_1$ is LI

(ii) To show $L(S_1) = V_3(\mathbb{R})$

W.K.T. $L(S_1) \subseteq V_3(\mathbb{R})$ —(1)

Let $\alpha \in V_3(\mathbb{R})$

i.e., $\alpha = (a,b,c) \in V_3(\mathbb{R})$

then $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$
$\in L(S_1)$

$\therefore \alpha \in L(S_1)$

$\therefore V_3(\mathbb{R}) \subseteq L(S_1)$ —(2)

$\therefore$ from (1) & (2) we have
$V_3(\mathbb{R}) = L(S_1)$.

$\therefore S_1$ is a basis of $V_3(\mathbb{R})$

(ii). $S_2 = \{(0,1,0), (0,0,1), (2,3,4)\} \subseteq V_3(\mathbb{R})$

Similar.

→ Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.

Sol: Let $V(F) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle/ a,b,c,d \in F \right\}$

Let $\alpha_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\alpha_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\alpha_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\alpha_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
be four elts of V.

Let $S = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \subseteq V$

(i) To show S is LI:

If $a_1, a_2, a_3, a_4 \in F$ then $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 = 0$

$\Rightarrow a_1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + a_2\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a_3\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + a_4\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_3 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a_1 = a_2 = a_3 = a_4 = 0$. $\therefore S$ is LI

w.k.T $L(S) \subseteq V$ — ①

Let $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 \in L(S).$$

$\therefore \alpha \in L(S)$

$\therefore V \subseteq L(S)$ — ②

$\therefore$ from ① & ② $V = L(S)$

$\therefore S$ is a basis of V.

Since the no. of elts in the basis 'S' is 4.

$\therefore \dim V = 4.$

→ Let V be the vector space of 2×2 matrices over F. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V s.t. $A_i^2 = A_i$ for each i.

Sol^n: $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in F \right\}$

Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

be any four vector elts of V s.t $A_i^2 = A_i$ for each

Let $S = \{A_1, A_2, A_3, A_4\} \subset V$.

(i) To show S LI

(ii) To show $L(S) = V$.

→ S.T the real field $\mathbb{R}$ is a vector space of infinite dimension over the rational field $\mathbb{Q}$.

Sol^n. We prove that the set $\{1, \pi, \pi^2, \ldots, \pi^n\}$ is LI over $\mathbb{Q}$ for any +ve integer 'n'

Suppose $a_0(1) + a_1(\pi) + a_2\pi^2 + \cdots + a_n\pi^n = 0$; where $a_i \in \mathbb{Q}$ and all a_i's are not zero,

Then π is a root of the non-zero polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ over $\mathbb{Q}$.

This is impossible, since π is a transcendental number.

$\therefore \{1, \pi, \pi^2, \ldots, \pi^n\}$ is LI over $\mathbb{Q}$ for all +ve integer 'n'.

Hence $\mathbb{R}$ is of an infinite dimension over $\mathbb{Q}$.

→ Determine whether or not the vectors $(1,-3,2)$, $(2,4,1)$ and $(1,1,1)$ form a basis of $\mathbb{R}^3$.

Solⁿ: W.K.T $\dim(\mathbb{R}^3) = 3$

if we show that the given three vectors are linearly independent they form a basis of $\mathbb{R}^3$.

Now form the matrix A, whose rows are given vectors. ~~and~~

$$\therefore A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(3) + 3(1) + 2(-2)$$
$$= 3 + 3 - 4$$
$$= 2 \neq 0$$

∴ The given vectors are L.I.

∴ They form a basis of $\mathbb{R}^3$.

→ Let V be vector space of ordered pairs of complex numbers over the field R. i.e, let V be the vector space $C(\mathbb{R})$

S.T the set $S = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis for V.

→ S.T the vectors $(1,0,-1)$, $(0,-3,2)$ and $(1,2,1)$ form a basis for the vector space $\mathbb{R}^3(\mathbb{R})$.

Solⁿ: W.K.T $\dim \mathbb{R}^3 = 3$

Now form the matrix A whose rows are given vectors.

$$\therefore A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

Now $|A| = 1(-7) - 0(\ ) - 1(3)$
$$= -7 - 3 = -10 \neq 0$$

∴ The given vectors are L.I.

∴ They form a basis for V(F).

→ S.T the set $\{(1, i, 0), (2i, 1, 1), (0, 1+i, 1-i)\}$ is a basis for $V_3(\mathbb{C})$.

Solⁿ: W.K.T $\dim V_3(\mathbb{C}) = 3$.

Now form the matrix whose rows are given vectors

$|\ \ | \neq 0$

$\mathbb{R}^3$.

Sol: W.K.T $\dim \mathbb{R}^3 = 3$

Form the matrix A whose rows are given set of vectors.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 0 \end{bmatrix}$$

$\therefore |A| = 0$

$\therefore$ The given set S is L.D.

$\therefore$ S cannot be a basis of $\mathbb{R}^3$.

→ S.T the set $S = \{(1,0,0), (1,1,0), (1,1,1), (0,1,0)\}$ is a spanning set of $\mathbb{R}^3$ but not a basis of $\mathbb{R}^3$.

Sol: To show $L(S) = \mathbb{R}^3$.

w.k.T $L(S) \subseteq \mathbb{R}^3$ — (I)

Let $(a,b,c) \in \mathbb{R}^3$ then

$(a,b,c) = x_1(1,0,0) + x_2(1,1,0) + x_3(1,1,1) + x_4(0,1,0)$

$= (x_1 + x_2 + x_3,\ x_2 + x_3 + x_4,\ x_3)$

$\Rightarrow x_1 + x_2 + x_3 = a$ — (1)

$x_2 + x_3 + x_4 = b$ — (2)

$x_3 = c$ — (3)

(2) ≡ $x_2 + x_4 = b - c$ — (4)

(1) ≡ $x_1 + x_2 = a - c$ — (5)

Take $x_2 = b$; $x_4 = -c$ in (4)

(5) ≡ $x_1 = a - b - c$

$\therefore (a,b,c) = (a-b-c)(1,0,0) + b(1,1,0) + c(1,1,1) - c(0,1,0)$

$\in L(S)$

$\therefore \mathbb{R}^3 \subseteq L(S)$ — (II)

from (I) & (II) $\mathbb{R}^3 = L(S)$

Since $\dim \mathbb{R}^3 = 3$ and S contains $4 = (3+1)$ vectors.

$\therefore$ S is L.D.

$\therefore$ S cannot be a basis of $\mathbb{R}^3$

→ Let $\{a, b, c\}$ be a basis for the vector space $\mathbb{R}^3$.
P.T the sets
$\{a+b, b+c, c+a\}$, $\{a, a+b, a+b+c\}$ are also bases of $\mathbb{R}^3$.

Sol^n Since $\{a, b, c\}$ is a basis of $\mathbb{R}^3$.
$\therefore \dim \mathbb{R}^3 = 3$

(i) Now let $x, y, z \in \mathbb{R}$ then
$$x(a+b) + y(b+c) + z(c+a) = 0$$
$$\therefore x = y = z = 0$$
$\therefore \{a+b, b+c, c+a\}$ is L.I
$\therefore$ It is a basis of $\mathbb{R}^3$.

(ii) Now let $x, y, z \in \mathbb{R}$ then
$$xa + y(a+b) + z(a+b+c) = 0$$

2006, 2011 → Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$. Hence find a basis for the subspace.

Theorem:-

Let $V(F)$ be a vector space and a subset $S=\{\alpha_1,\alpha_2,\ldots\alpha_n\}$ of $V(F)$ (i.e, $S\subseteq V$) be a linearly independent set. If $\alpha\in V(F)$ and $\alpha\notin L(S)$ then show that $S_1=\{\alpha,\alpha_1,\alpha_2\ldots\alpha_n\}$ is a linearly independent set

(or)

If $S=\{\alpha_1,\alpha_2,\ldots\alpha_n\}$ is a L.I set of vectors in V and $\alpha\in V$ is such that $\alpha\notin L(S)$, then $\{\alpha,\alpha_1,\alpha_2,\ldots\alpha_n\}$ is L.I set.

proof: Let $a, a_1, a_2, \ldots a_n \in F$

then $a\alpha+a_1\alpha_1+a_2\alpha_2+\cdots\cdots+a_n\alpha_n=0$ —— (1)

If $a\neq 0$ then

$$\alpha=\left(\frac{-a_1}{a}\right)\alpha_1+\left(\frac{-a_2}{a}\right)\alpha_2+\cdots\cdots+\left(\frac{-a_n}{a}\right)\alpha_n$$

$\in L(S)$

$\Rightarrow \alpha\in L(S)$ which is a contradiction to the hypothesis that $\alpha\notin L(S)$

$\therefore a=0$

(1) $\equiv a_1\alpha_1+a_2\alpha_2+\cdots\cdots+a_n\alpha_n=0$

$\Rightarrow a_1=a_2=\cdots=a_n=0$ $(\because S$ is L.I$)$

$\Rightarrow a=a_1=a_2=\cdots\cdots=a_n=0$

$\therefore S_1=\{\alpha,\alpha_1,\alpha_2,\ldots\alpha_n\}$ is L.I set.

problem: Extend the set $\{(1,1,1),(1,0,0)\}$ to form a basis of $\mathbb{R}^3$.

Sol^n: Method(I)

Let $\mathbb{R}^3=\{(x,y,z)/x,y,z\in\mathbb{R}\}$

Let $S=\{(1,1,1),(1,0,0)\}\subseteq\mathbb{R}^3$.

Let $a,b\in\mathbb{R}$ then

$a(1,1,1)+b(1,0,0)=(0,0,0)$

$\Rightarrow a+b=0 \Rightarrow \boxed{b=0}$

$\boxed{a=0}$

$\therefore a=b=0$

$\therefore S$ is L.I.

Now $L(S)=\{a(1,1,1)+b(1,0,0)/a,b\in\mathbb{R}\}$

$=\{(a+b,a,a)/a,b\in\mathbb{R}\}$

Let $\alpha=(0,0,1)\in V$ then $\alpha\notin L(S)$.

$\therefore$ The set $S'=\{(1,1,1),(1,0,0)\ (0,0,1)\}$ is L.I.

$\therefore S^1$ is a basis of $\mathbb{R}^3$.

Similarly $(0,1,0) \notin L(S)$.

$\therefore$ The set $\{(1,1,1),(1,0,0),(0,1,0)\}$ is LI set.

$\therefore$ It is also basis of $\mathbb{R}^3$.

Method II

$\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$

$S = \{(1,1,1),(1,0,0)\}$ LI set

Since $a(1,1,1) + b(1,0,0) = (0,0,0)$ where $a,b \in \mathbb{R}$

$\Rightarrow (a+b, a, a) = (0,0,0)$

$\Rightarrow a = b = 0$

$\therefore S$ is LI

W.K.T the vectors

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ form a standard basis of $\mathbb{R}^3$.

$\therefore$ The vectors $\alpha = (1,1,1)$, $\beta = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ span $\mathbb{R}^3$

but any basis of $\mathbb{R}^3$ contains exactly 3 LI elts.

Let us check whether α, β, e_2 are LI or not:

Now form the matrix A whose rows the vectors α, β, e_2

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow |A| = 1(-1) - 1(0) = -1 \neq 0$$

$\therefore \alpha, \beta, e_2$ are LI vectors.

$\therefore$ The vectors form a basis.

Similarly the set $\{\alpha, \beta, e_3\}$ is also LI.

$\therefore$ It is a basis of $\mathbb{R}^3$.

Method III. Let $S = \{(1,1,1),(1,0,0)\} \subseteq \mathbb{R}^3$

Since $a(1,1,1) + b(1,0,0) = (0,0,0)$ where $a,b \in \mathbb{R}$

$\Rightarrow a = 0, b = 0$

$\therefore S$ is LI

W.K.T the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, & $e_3 = (0,0,1)$ form a standard basis of $\mathbb{R}^3$.

$e_3 = (1,1,1)$ span $\mathbb{R}^3$.

but any basis of $\mathbb{R}^3$ contains exactly 3 LI elts.

Let us check whether α, β, e_2 are LI or not :-

Now form the matrix A whose rows are the vectors α, β, e_2; reduce it to echelon matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{R_2 \to R_2 - R_1} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 + R_2} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ The echelon matrix of A has no zero rows.

$\therefore$ The vectors α, β, e_2 are LI.

$\therefore$ They form a basis of $\mathbb{R}^3$.

and also the vectors $(1,1,1)$, $(0,-1,-1)$ and $(0,0,-1)$ are LI. ($\because$ The non-zero rows of a matrix are LI)

$\therefore$ These are also form a basis of $\mathbb{R}^3$.

Note: The extension of linearly independent vectors to a basis is not unique.

$\rightarrow$ Extend the set $\{(0,0,0,1), (1,1,0,0), (0,1,-1,0)\}$ to form a basis of $\mathbb{R}^4$.

$\rightarrow$ Extend the set $S = \{(1,1,0)\}$ to form two different bases of $\mathbb{R}^3$.

Sol: Since $(1,1,0) \neq (0,0,0)$

$\therefore S$ is LI set.

and $L(S) = \{a(1,1,0) / a \in \mathbb{R}\}$

$= \{(a,a,0) / a \in \mathbb{R}\}$

Since $(0,0,1) \notin L(S)$

$\therefore S_1 = \{(1,1,0), (0,0,1)\}$ is LI.

Now $L(S_1) = \{a(1,1,0) + b(0,0,1) / a,b \in \mathbb{R}\}$

$= \{(a+b, a, b) / a,b \in \mathbb{R}\}$

Since $(0,1,1) \notin L(S)$

$S_2 = \{(1,1,0), (0,0,1), (0,1,1)\}$ is LI

$\therefore S_2$ is a basis of $\mathbb{R}^3$

Similarly $\{(1,1,0), (0,0,1), (0,2,0)\}$ is also basis of $\mathbb{R}^3$.

→ Extend the set $\{(3,-1,2)\}$ to two different bases for $\mathbb{R}^3$.

→ Can the set $\{(1,0,0,0), (0,1,0,0), (1,-1,0,0)\}$ be extended to form a basis of $\mathbb{R}^4$?

Sol: The given set of vectors are not L.I vectors.

Since $1(1,0,0,0) + (-1)(0,1,0,0) + (-1)(1,-1,0,0) = (0,0,0,0)$

$\therefore$ The given set of vectors cannot be extend to form a basis

→ Determine whether or not the following vectors form a basis.

(i) $(1,-1,0), (1,3,-1), (5,3,-2)$ of $\mathbb{R}^3(\mathbb{R})$

~~(ii) $(1,0,1), (1,1,0)$~~, $(1,1,-1)$ of $\mathbb{R}^3(\mathbb{R})$

(iii) $(6,2,3,4), (0,5,-3,1), (0,0,7,-2), (0,0,0,4)$ of $\mathbb{R}^4(\mathbb{R})$

(iv) $(1,-2,4,1), (2,-3,9,-1), (1,0,6,-5), (2,-5,7,5)$ of $\mathbb{R}^4(\mathbb{R})$

→ Given two linearly independent vectors $(1,0,1,0)$ and $(0,-1,1,0)$ of $\mathbb{R}^4$, find a basis of $\mathbb{R}^4$ which includes these two vectors.

→ Let V be the vectorspace of all 2×2 symmetric matrices over $\mathbb{R}$. Find a basis and the dimension of V.

Sol: $V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} \middle/ a,b,c \in \mathbb{R} \right\}$ $\quad$ ($\because A^T = A$, A is symmetric)

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq V$

① To show S is LI.

Let $x,y,z \in \mathbb{R}$ then

$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow x = y = z = 0$

W.K.T $L(S) \subseteq V$ — (i)

Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\in L(S)$.

$\therefore V \subseteq L(S)$ — (ii)

$\therefore$ from (i) & (ii) $L(S) = V$.

$\therefore$ S is a basis of V.

$\therefore \dim V = 3$

→ Let V be the vector space of 3×3 symmetric matrices over F. then show that dim V = 6 by exhibiting a basis of V.

Soln: Let $V = \left\{ \begin{bmatrix} a & h & g \\ h & b & e \\ g & e & c \end{bmatrix} \middle| a, b, c, h, g, e \in F \right\}$

Let $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \subseteq V.$

Note: → Dimension of the vectorspace V of all 2×2 symmetric matrices is $3 = 2+1$

→ Dimension of the vectorspace V of all 3×3 symmetric matrice is $6 = 3+2+1$

→ Dimension of the vectorspace V of all n×n symmetric matrice is $n + (n-1) + (n-2) + \cdots + 3 + 2 + 1$.

$= \frac{n(n+1)}{2}$.

→ V be the vector space of 2×2 antisymmetric matrices over F. Show that dim V = 1 by exhibiting a basis of V.

Soln Let $V = \left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \middle/ a \in F \right\}$

Let $S = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \subseteq V.$

→ Let V be the vector space of 3×3-antisymmetric matrices over F. Show that dim V = 3 by exhibiting a basis of V.

Soln Let $V = \left\{ \begin{bmatrix} 0 & h & g \\ -h & 0 & e \\ -g & -e & 0 \end{bmatrix} \middle/ h, g, e \in F \right\}$

Let $S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \subseteq V.$

Note: → The dimension of the vectorspace of 2×2 skew symmetric matrices over F is $1 (= 2-1)$

→ The dimension of the vector space of 3×3 skew symmetric matrices over F is $3 \left(= (3-1)+(3-2)\right)$

→ The dimension of the vector space of $n\times n$ skew symmetric matrices over F is $(n-1)+(n-2)+\cdots\cdots+2+1$

$$= \frac{n(n-1)}{2}$$

Note: Let V be the vector space of $m\times n$ matrices over a field F.

Let $E_{ij} \in V$ be the matrix with 1 as ij-entry and elsewhere zero

then the set $\{E_{ij}\}$ is a basis of V and $\dim V = mn$.

(This basis is called the standard basis of V).

→ Let $V(\mathbb{R})$ be the real vectorspace of all 2×3 matrices with real entries. Find a basis for $V(\mathbb{R})$. What is the dimension of $V(\mathbb{R})$.

Sol^n Let $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\} \subset V(\mathbb{R})$

are L.I. and $L(S) = V$.

$\therefore$ S is a basis of V.

$\therefore \dim V = 6$

→ Let V be the set of all real valued functions $y=f(x)$ satisfying $\frac{d^2y}{dx^2} + 4y = 0$. Prove that V is a 2-dimensional real vector space.

Sol^n $\frac{d^2y}{dx^2} + 4y = 0 \Rightarrow (D^2+4)y = 0$ where $D = \frac{d}{dx}$ —— ①

A.E of ① is $m^2+4=0$

$\Rightarrow m = \pm 2i$

$\therefore$ G.S. of ① is $y = c_1\cos 2x + c_2 \sin 2x$ —— ②

where c_1 and c_2 are any real constants.

$\therefore$ Since V is the set of all real valued functions $y=f(x)$ satisfying $\frac{d^2y}{dx^2}+4y=0$

$\therefore V = \{ y = c_1\cos 2x + c_2 \sin 2x \mid c_1, c_2 \in \mathbb{R}\}$ is a vectorspace.

(Here we must show V is a vectorspace).

Let $S = \{\cos 2x, \sin 2x\} \subseteq V$.

The wronskian of $y_1(x) = \cos 2x$, $y_2(x) = \sin 2x$

is $W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

$| -2\sin x \quad 2\cos x |$

$\therefore$ S is a LI subset of V.

By ② $L(S) = V$.

$\therefore$ S is a basis of V over $\mathbb{R}$

$\therefore$ V is a two dimensional real vector space

→ Let V be the set of all real-valued functions $y = f(x)$ satisfying $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.

S.T $V(\mathbb{R})$ is a 3-dimensional real vector space. write down a basis of this vector space.

Sol: $m^3 - 7m - 6 = 0 \Rightarrow (m+1)(m^2 - m - 6) = 0$

$\Rightarrow (m+1)(m-3)(m+2) = 0$

$\Rightarrow m = -1, -2, 3$

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$m = -1, -2, -3$

→ S.T the set of all real valued continuous functions $y = f(x)$ satisfying the differential equation $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ is a vectorspace over $\mathbb{R}$.

Give a basis for the vectorspace.

→ S.T the matrices $\begin{bmatrix} 1 & 5 \\ 5 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$ form a basis of $V(\mathbb{R})$.

where V is the vector space of all 2×2 symmetric matrices over reals.

→ S.T the dimension of the vector space $\mathbb{Q}(\sqrt{2})$ over $\mathbb{Q}$ is 2.

Sol: Let $\mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} / a, b \in \mathbb{Q} \}$.

Let $S = \{ 1, \sqrt{2} \} \subseteq \mathbb{Q}\sqrt{2}$

(i) To show S is LI

Let $x, y \in \mathbb{Q}$ then

$x(1) + y(\sqrt{2}) = 0 + 0\sqrt{2}$

$\Rightarrow x = y = 0$

(ii) $L(S) = \mathbb{Q}(\sqrt{2})$.

W.K.T $L(S) \subseteq \mathbb{Q}\sqrt{2}$ —(1)

and let $a + b\sqrt{2} \in \mathbb{Q}\sqrt{2}$ then $a + b\sqrt{2} = a(1) + b\sqrt{2} \in L(S)$

$\therefore \mathbb{Q}(\sqrt{2}) \subseteq L(S)$. —②

$\therefore$ from (1) & (2) $L(S) = \mathbb{Q}(\sqrt{2})$.

$\therefore$ S is a basis and $\dim(\mathbb{Q}\sqrt{2}) = 2$

→ S.T the dimension of vector space $Q(\sqrt{2},\sqrt{3})$ over Q is 4.

Sol: Let $Q(\sqrt{2},\sqrt{3}) = \{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3} / a,b,c,d \in Q\}$.

Let $S = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$

→ S.T $f_1(t) = 1$, $f_2(t) = t-2$, $f_3(t) = (t-2)^2$ form a basis of P_3, the space of polynomial with degree ≤ 2.

Express $3t^2-5t+4$ as a l.c. of f_1, f_2, f_3.

Sol: Let $f(t) = 3t^2-5t+4 \in P_3$.

then $f(t) = xf_1(t)+yf_2(t)+zf_3(t)$ where $x,y,z \in F$.

$\Rightarrow 3t^2-5t+4 = x(1)+y(t-2)+z(t-2)^2$ —— ①

$= x+ty-2y+t^2z+4z-4tz$.

$\Rightarrow 3t^2-5t+4 = zt^2+(y-4z)t+(x-2y+4z)$

$\Rightarrow \boxed{z=3}$

$y-4z=-5$

$\Rightarrow \boxed{y=7}$

$x-2y+4z=4$

$\Rightarrow \boxed{x=6}$

$\therefore$ ① $\equiv$ $3t^2-5t+4 = 6(1)+7(t-2)+3(t-2)^2$

$=$ l.c. of f_1, f_2, f_3

Dimension of a Subspace:-

Theorem: If W is a subspace of a finite dimensional vector space $V(F)$ then W is finite dimensional and $\dim W \leq \dim V$.

Further $V=W \Leftrightarrow \dim V = \dim W$.

proof: Given that W is a subspace of finite dimensional vector space $V(F)$.

Let $\dim V = n$

(i) To prove W is finite dimensional.

If possible suppose that W is not finite dimensional.

then W ~~is~~ has infinite basis.

Take S_1 is an infinite basis of W.

$\therefore S_1$ is LI in W.

But S_1 is the infinite ...

$\therefore S_1$ is a LI subset of V having more than 'n' elts.

which is contradiction.

$\therefore$ our supposition is wrong.

$\therefore$ W is a finite dimensional.

Take $\dim W = m$.

Now we have to S.T $m \le n$.

Let $S_1 = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ be a basis of W.

$\Rightarrow S_1$ is LI set in W.

$\Rightarrow S_1$ is LI set in V.

Any LI subset of vector space $V(F)$ can be extended to form a basis of V.

$\therefore$ there exists a basis S of V s.t $S_1 \subseteq S$.

$\Rightarrow$ No. of elts in $S_1 \le$ No. of elts in S.

$\Rightarrow m \le n$

i.e, $\dim W \le \dim V$.

(ii) If $V = W$ then

W is a subspace of V and

V is " " W

$\therefore \dim W \le \dim V$ & $\dim V \le \dim W$.

$\Rightarrow \dim V = \dim W$.

Conversely suppose that $\dim V = \dim W$.

Let $\dim V = \dim W = n$ (say)

Let S be a basis of W.

then $L(S) = W$ and S has 'n' LI vectors.

Also S is subset of V ($\because S \subset W \subset V$)

and S has n LI vectors (i.e, S is LI in V)

$\Rightarrow S$ is a basis of V.

$\Rightarrow L(S) = V$.

$\therefore \underline{\underline{V = W}}$

Note [1]. If $W=\{0\}$ then the dimension $W=0$.

[2]. If W is a proper subspace of a finite-dimensional vectorspace V, then W is finite dimensional and $\dim W < \dim V$.

[3]. If V is finite dimensional and W is a subspace of V such that $\dim V = \dim W$. then $V=W$.

Th. If W_1, W_2 are two subspaces of a finite dimensional vectorspace $V(F)$, then $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

proof Given that W_1 & W_2 are two subspaces of $V(F)$.

$\therefore W_1+W_2, W_1 \cap W_2$ are also subspaces of $V(F)$.

Since W_1, W_2, W_1+W_2 & $W_1 \cap W_2$ are subspaces of finite dimensional vector space $V(F)$.

$\therefore W_1, W_2, W_1+W_2$ & $W_1 \cap W_2$ are all finite dimensional.

Let $\dim(W_1 \cap W_2) = K$ and

Let $S = \{\gamma_1, \gamma_2, \ldots \gamma_K\} \subseteq W_1 \cap W_2$ be a basis of $W_1 \cap W_2$.

then $S \subseteq W_1$ and $S \subseteq W_2$

Since S is LI and $S \subseteq W_1$

$\therefore S$ can be extended to form a basis of W_1.

Let $S_1 = \{\gamma_1, \gamma_2, \ldots \gamma_K, \alpha_1, \alpha_2, \ldots \alpha_m\}$ be a basis of W_1.

$\therefore \dim(W_1) = k+m$

Since S is LI and $S \subseteq W_2$.

$\therefore S$ can be extended to form a basis of W_2.

Let $S_2 = \{\gamma_1, \gamma_2, \ldots \gamma_K, \beta_1, \beta_2, \ldots \beta_t\}$ be a basis of W_2

$\therefore \dim(W_2) = k+t$

$\therefore \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = (k+m)+(k+t)-k$

$= k+m+t$.

Now we have to show that $\dim(W_1+W_2) = k+m+t$.

For this we have to show that the set

$S_3 = \{\gamma_1, \gamma_2, \ldots \gamma_K, \alpha_1, \alpha_2, \ldots \alpha_m, \beta_1, \beta_2, \ldots \beta_t\}$ is a basis of W_1+W_2.

Let $c_1, c_2, \dots, c_k, a_1, a_2, \dots a_m, b_1, b_2, \dots b_t$ then

$$(c_1\gamma_1+c_2\gamma_2+\cdots+c_k\gamma_k)+a_1\alpha_1+a_2\alpha_2+\cdots+a_m\alpha_m+b_1\beta_1+b_2\beta_2+\cdots+b_t\beta_t=0 \quad (1)$$

$$\Rightarrow b_1\beta_1+b_2\beta_2+\cdots+b_t\beta_t=-(c_1\gamma_1+c_2\gamma_2+\cdots+c_k\gamma_k+a_1\alpha_1+a_2\alpha_2+\cdots+a_m\alpha_m) \quad (2)$$

Now $-(c_1\gamma_1+c_2\gamma_2+\cdots+c_k\gamma_k+a_1\alpha_1+a_2\alpha_2+\cdots+a_m\alpha_m)\in W_1$ (3) ($\because$ it is l.c. of S_1)

and $b_1\beta_1+b_2\beta_2+\cdots+b_t\beta_t\in W_2$ — (4) ($\because$ it is a l.c. of elts of S_2)

(2) $\Rightarrow b_1\beta_1+b_2\beta_2+\cdots+b_t\beta_t\in W_1$ (5) (by (3))

$\therefore$ from (4) & (5) we have

$b_1\beta_1+b_2\beta_2+\cdots+b_t\beta_t\in W_1\cap W_2$

$\therefore$ It can be expressed as a l.c of the basis of $W_1\cap W_2$.

$\therefore$ we have $b_1\beta_1+b_2\beta_2+\cdots+b_t\beta_t=d_1\gamma_1+d_2\gamma_2+\cdots+d_k\gamma_k$

$$\Rightarrow b_1\beta_1+b_2\beta_2+\cdots+b_t\beta_t-d_1\gamma_1-d_2\gamma_2-\cdots-d_k\gamma_k=0$$

Since S_2 is LI set.

$\therefore b_1=b_2=\cdots=b_t=d_1=d_2=\cdots=d_k=0$

$\therefore$ (1) $\Rightarrow c_1\gamma_1+c_2\gamma_2+\cdots+c_k\gamma_k+a_1\alpha_1+a_2\alpha_2+\cdots+a_m\alpha_m=0$

Since S_1 is LI.

$\therefore c_1=c_2=c_3=\cdots=c_k=a_1=a_2=\cdots=a_m=0$

$\therefore c_1=c_2=c_3=\cdots=c_k=a_1=a_2=\cdots=a_m=b_1=b_2=\cdots=b_t=0$

$\therefore S_3$ is LI set.

(ii) To show $L(S_3)=W_1+W_2$ — (A)

W.K.T $L(S_3)\subseteq W_1+W_2$

Let $\alpha\in W_1+W_2$ then

$\alpha=\alpha_i+\alpha_j$ where $\alpha_i\in W_1$ and $\alpha_j\in W_2$.

Since α_i is a l.c. of the basis of W_1 and α_j is a l.c of the basis of W_2

$\therefore \alpha=$ l.c of the basis of W_1+ l.c. of the basis of W_2

$= $ l.c. of elts of S_3

$\in L(S_3)$

$\therefore$ If $\alpha \in W_1 + W_2$ then $\alpha \in L(S_3)$

$\therefore W_1 + W_2 \subseteq L(S_3)$

from (A) & (B) we have

$L(S_3) = W_1 + W_2$.

$\therefore S_3$ is a basis of $W_1 + W_2$

$\therefore \dim(W_1 + W_2) = k + m + t$.

$\therefore \dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Note: (1) If W_1 and W_2 are two subspaces of a FDVS $V(F)$ such that $W_1 \cap W_2 = \{0\}$ then $\dim(W_1 + W_2) = \dim W_1 + \dim W_2$.

Defn Row-equivalence of two matrices:

A matrix A is said to be row-equivalent to a matrix B iff B can be obtained from A by a finite no. of elementary row operations.

Defn Coloumn-equivalence of two matrices:

A matrix A is said to be coloumn equivalent to a matrix B iff B can be obtained from A by a finite no. of elementary coloumn operations.

Note: Elementary row operations are:

(i) Interchange of the i^{th} & j^{th} rows: $R_i \leftrightarrow R_j$

(ii) Multiplying the i^{th} row by a non-zero scalar k: $R_i \to kR_i$

(iii) Adding to the i^{th} row k times the j^{th} row: $R_i \to R_i + kR_j$

Defn Row space of a matrix:

Let $A = [a_{ij}]$ be an $m \times n$ matrix over a field F

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$R_1 = (a_{11}, a_{12}, \dots a_{1n})$, $R_2 = (a_{21}, a_{22} \dots)$ $\dots \dots R_m = (a_{m1}, a_{m2} \dots a_{mn})$ as vectors in F^n or $V_n(F)$.

($\because$ each of these being an n-tuple over F).

The linear span of these vectors i.e., $L(\{R_1, R_2, \dots R_m\})$ is a subspace of F^n and is called the row space of A

i.e, row $sp(A) = \text{Span}(R_1 R_2 \dots R_m)$

Similarly, the space spanned by the coloumn vectors i.e, $L(\{C_1, C_2, \dots C_n\})$ is a subspace of F^m and is called the coloumn space of A.

where $C_1 = (a_{11}\ a_{21}, a_{31} \dots a_{m1})$

$C_2 = (a_{12}\ a_{22}\ a_{32} \dots a_{m2})$

$\vdots$

$C_n = (a_{1n}, a_{2n}, \dots \dots a_{mn})$

i.e, $\text{Col sp}(A) = \text{Span}(C_1, C_2, \dots C_n)$.

Note: [1]. Coloumn space of A is the same as the row space of A^T.

i.e, $\text{Col sp}(A) = \text{row sp}(A^T)$.

Theorem: Row equivalent matrices have the same row space

proof: Let A and B be two row equivalent matrices — Then by definition of row equivalence, each row of B is either a row of A or l.c. of rows of A.

$\therefore$ The row space of B is contained in the row space of A.

Sly applying the inverse elementary row operation B and obtain A

$\therefore$ The row space of A is contained in the row space of B.

$\therefore$ The row spaces of A & B are same.

Note [1]. Coloumn equivalent matrices have the same Coloumn space.

[2]. Let A and B be two row-reduced echelon matrices. Then A and B have the same row space iff they have the same non-zero rows

→ Determine whether the following matrices have the same row space.

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix},\quad B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix},\quad C = \begin{pmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{pmatrix}.$$

Sol^n: The matrices have the same row space iff their row reduced echelon matrices have the same non-zero rows.

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\sim \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix}$$

$R_1 \to R_1 - R_2$

$$\sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1$

$$\sim \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{pmatrix}$$

$R_2 \to R_2 - 4R_1$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\therefore$ A and C have the same row space and B has different row space.

Coloumn space.

$$A=\begin{pmatrix}1&3&5\\1&4&3\\1&1&9\end{pmatrix};\quad B=\begin{pmatrix}1&-2&3\\-2&-3&-4\\7&12&17\end{pmatrix}$$

Sol^n: A and B have the same coloumn space iff A^T & B^T have same row space.

Now A^T & B^T reduce to row canonical form

$$A^T=\begin{pmatrix}1&1&1\\3&4&1\\5&3&9\end{pmatrix} \qquad B^T=\begin{pmatrix}1&-2&7\\2&-3&12\\3&-4&17\end{pmatrix}$$

$R_2\to R_2-3R_1,\ R_3\to R_3-5R_1$ $\qquad$ $R_2\to R_2-2R_1,\ R_3\to R_3-3R_1$

$$\sim\begin{pmatrix}1&1&1\\0&1&-2\\0&-2&4\end{pmatrix} \qquad \sim\begin{pmatrix}1&-2&7\\0&1&-2\\0&2&-4\end{pmatrix}$$

$R_3\to R_3+2R_2$ $\qquad$ $R_1\to R_1+R_3$

$$\sim\begin{pmatrix}1&1&1\\0&1&-2\\0&0&0\end{pmatrix} \qquad \sim\begin{pmatrix}1&0&-1\\0&1&-2\\0&2&-4\end{pmatrix}$$

$R_1\to R_1-R_2$ $\qquad$ $R_3\to R_3-2R_2$

$$\sim\begin{pmatrix}1&0&-1\\0&1&-2\\0&0&0\end{pmatrix} \qquad \sim\begin{pmatrix}1&0&-1\\0&1&-2\\0&0&0\end{pmatrix}$$

$\therefore$ A^T & B^T have the same row space.

$\therefore$ A and B have the same coloumn space.

Note: As the non-zero rows of an echelon matrix are LI and row equivalent matrices have same row space it follows that

$\therefore$ Dimension of row space of A (i.e. dimension of subspace) = Maximum no. of LI rows in A.

= maximum no. of LI rows of echelon matrix of A

i.e. = no. of non-zero rows of echelon matrix of A

→ Reduce the matrix $A=\begin{bmatrix}0&1&-3&-1\\2&-1&4&0\\4&1&-1&-3\end{bmatrix}$ to row-reduced Echelon form. Also find a basis for the row space and its dimension.

Sol^n $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{bmatrix}$

$R_2 \to R_1$

$$\sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 4 & 1 & -1 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - 2R_1} \sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 3 & -9 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \sim \begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \to \frac{1}{2}R_1} \sim \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the row reduced echelon form.

$\therefore$ A basis for the row space is $\left\{\left(1, 0, \frac{1}{2}, -\frac{1}{2}\right), (0, 1, -3, -1)\right\}$ and the dimension of row space is 2.

Let $U = \text{Span}(u_1, u_2, u_3)$ and $W = \text{span}(v_1, v_2)$ be two subspaces of $\mathbb{R}^4$ where $u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, 1, -2)$, $u_3 = (3, 6, 3, -7)$, $v_1 = (1, 2, -4, 11)$, $v_2 = (2, 4, -5, 14)$; S.T. $U = W$.

Sol^n: form the matrix A whose rows are u_i's $(i = 1, 2, 3)$ and reduce it to row reduced echelon form.

$$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2,\ R_2 \to \frac{R_2}{3}} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim$$

$$\xrightarrow{R_1 \to R_1 + R_2} \sim \begin{pmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{pmatrix}$$

Now form the matrix whose rows are v_i's $(i = 1, 2)$ and reduce it to row reduced echelon form.

$$B = \begin{pmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \sim \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_2} \sim \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8/3 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 4R_2} \sim \begin{pmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{pmatrix}$$

Since the non-zero rows of the row reduced matrices are same,

$\therefore$ The row spaces of A & B are equal.

$\therefore U = V$.

generated by $(1,-4,-2,1)$, $(1,-3,-1,2)$, $(3,-8,-2,7)$.
Also extend the basis of W to a basis of the whole space $\mathbb{R}^4$.

Sol: Now form the matrix A whose rows are the given vectors and reduce it to echelon form.

$$A = \begin{bmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The non-zero rows in the echelon matrix
$(1,-4,-2,1)$ and $(0,1,1,1)$ form a basis of W.
and $\dim W = 2$

In particular, the original three given vectors are L.D.
Since $\mathbb{R}^4$ is 4-dimensional vector space.
$\therefore$ we require for L.I vectors which include the above two vectors.
$\therefore$ The vectors $(1,-4,-2,1)$, $(0,1,1,1)$, $(0,0,1,0)$, and $(0,0,0,1)$ are L.I over $\mathbb{R}$. (Since they form an echelon matrix)
$\therefore$ These vectors form a basis of $\mathbb{R}^4$.
$\therefore$ It is an extension of the basis of W.

2004 → Let S be the space generated by vectors $\{(0,2,6), (3,1,6), (4,-2,-2)\}$. What is the dimension of the space S? Find basis for S.

1985 → Consider the basis $S = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ where $v_1 = (1,1,1)$, $v_2 = (1,1,0)$, $v_3 = (1,0,0)$.
Express $(2,-3,5)$ in terms of the basis v_1, v_2, v_3.

→ Let W be the subspace of $\mathbb{R}^5$ spanned by $u_1 = (1,2,-1,3,4)$ and $u_2 = (2,4,-2,6,8)$, $u_3 = (1,3,2,2,6)$, $u_4 = (1,4,5,1,8)$ and $u_5 = (2,7,3,3,9)$. Find a subset of the vectors which form a basis of W.

Sol: Method 1 Let $\{u_1, u_2, u_3, u_4, u_5\}$ which spans W.
Since $u_2 = 2u_1$
$\therefore u_1$ & u_2 are L.D.

$\therefore$ Eliminate the vector u_2 from S.

$\therefore$ if $S_1 = \{u_1, u_3, u_4, u_5\}$ then subspace W of $\mathbb{R}^5$ spanned by S_1.

Now there exists no real number c s.t. $u_3 = cu_1$

$\therefore$ u_3, u_1 are L.I.

Sly $u_4 \neq cu_1$ & $u_5 \neq cu_1$.

Now let us check whether the vector u_4 is a l.c. of u_1, u_3, u_5 or not.

Let $u_4 = au_1 + bu_3 + cu_5$ where $a, b, c \in \mathbb{R}$

$u_4 = (1,4,5,1,8) = 1(1,2,-1,3,4) + 2(1,3,2,2,6) - 0(2,7,3,3,9)$

$\therefore$ u_4 is l.c. of u_1, u_3 and u_5.

$\therefore$ S_1 is LD.

$\therefore$ Eliminate the vector u_4 from S_1.

if $S_2 = \{u_1, u_3, u_5\}$ then subspace W of $\mathbb{R}^5$ spanned by S_2.

No vector of S_2 is a l.c of others.

$\therefore$ S_2 is LI subset of S.

$\therefore$ S_2 is a basis of W

Method 2: Form the matrix A whose rows are given vectors and reduce the matrix to an echelon form but with interchanging any zero rows.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix} \underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1}}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3,\ R_5 \to R_5 - 3R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

$\therefore$ The non-zero rows are the first, third and fifth rows.

$\therefore$ u_1, u_3, u_5 form a basis of W

without changing any zeros. $v_1 = (2,-2,4,4)$, $v_2 = (1,9,3)$, $v_3 = (-2,-4,1)$ and $v_4 = (3,7,-$

1985. Determine a basis of the subspace spanned by the vectors $v_1 = (1,2,3)$, $v_2 = (2,1,-1)$, $v_3 = (1,-1,-4)$, $v_4 = (4,2,-2)$.

→ Let V_1 and V_2 be the subspaces of R^4 generated by $\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$ respectively. find the dimension of (i) V_1 (ii) V_2 (iii) V_1+V_2 (iv) $V_1 \cap V_2$.

Sol

Sol^n Let $S_1 = \{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and

$S_2 = \{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$.

(i) form the matrix A whose rows are the vectors of S_1 and reduce it to an echelon matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ The echelon matrix of A has two non-zero rows.

∴ $\{(1,1,0,-1), (0,1,3,1)\}$ form a basis of V_1.

∴ $\dim V_1 = 2$

(ii) proceed as in (i) ∴

$\dim V_2 = 2$

(iii) Since V_1 and V_2 are two subspaces of $\mathbb{R}^4$.

∴ V_1+V_2 is also subspace of $\mathbb{R}^4$.

∴ V_1+V_2 is the space generated by all the six vectors. (i.e., $S_1 \cup S_2$)

Now form the matrix A whose rows are the given six vectors and reduce it to an echelon form.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1,\ R_6 \to R_6 - R_1}} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_5 \to R_5 - R_4$
$R_6 \to R_6 - 2R_4$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[R_3 \to R_3 - R_2]{R_3 \leftrightarrow R_4} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ The echelon matrix of A has three non-zero rows.

$\therefore \dim(V_1 + V_2) = 3$

(iv) $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2)$
$= 2 + 2 - 3$
$= 1$

1996 → In R^4, let W_1 be the space generated by $(1, 4, 0, -1)$ $(2, 4, 6, 0)$ and $(-2, -3, -3, 1)$ and let W_2 be the space generated by $(-1, -2, -2, 2)$, $(4, 6, 4, -6)$ $(1, 3, 4, -3)$.
Find a basis for the space $W_1 + W_2$.

→ Let $V = \mathbb{R}^4(\mathbb{R})$
$W = \{(a, b, c, d) \in \mathbb{R}^4 / a = b + c,\ c = b + d\}$.
Find a basis and the dimension of W.

Sol^n: Let $\alpha_1 = (1, 1, 0, -1)$ and $\alpha_2 = (0, 1, -1, -2)$
then $\alpha_1, \alpha_2 \in W$ and are L.I.

Since $x\alpha_1 + y\alpha_2 = 0$ where $x, y \in \mathbb{R}$.

$\Rightarrow x(1, 1, 0, -1) + y(0, 1, -1, -2) = 0$
$\Rightarrow (x, x+y, -y, -x-2y) = (0, 0, 0, 0)$
$\Rightarrow x = 0 = y$.

To show W is spanned by $\alpha_1 \& \alpha_2$.

Let $(a, b, c, d) \in W$ then $a = b + c$ & $c = b + d$ — (1)

Since $a(1, 1, 0, -1) + c(0, 1, -1, -2)$
$= (a, a - c, c, -a + 2c)$
$= (a, b, c, d)$ by (1).
$\therefore W$ is spanned by $\{\alpha_1, \alpha_2\}$.
$\{\alpha_1, \alpha_2\}$ is a basis of W and $\dim W = 2$.

(OR) $W = \{(a, b, c, d) / a = b + c,\ c = b + d\} \subseteq R^4$

Let $\alpha = (a, b, c, d)$.
$\therefore \alpha = (b + c, b, b + d, d)$ where $a = b + c$, $c = b + d$
$= (2b + d, b, b + d, d)$
$= b(2, 1, 1, 0) +$
$d(1, 0, 1, 1)$

$W = L(\{\alpha_1, \alpha_2\})$
where $\alpha_1 = (2, 1, 1, 0)$
$\alpha_2 = (1, 0, 1, 1)$
are L.I.

$\therefore \{\alpha_1, \alpha_2\}$ is basis of W.

be two subspaces of $V = \mathbb{R}^3(\mathbb{R})$.
find the dimension of $A+B$.

Sol^n Let $(x, y, 0) \in A$ then
$(x, y, 0) = x(1,0,0) + y(0,1,0)$
$\therefore A = L(\{e_1, e_2\})$ where $e_1 = (1,0,0)$
$e_2 = (0,1,0)$ are LI
$\therefore$ The set $\{e_1, e_2\}$ is basis of A
$\therefore \dim A = 2$

Let $(0, y, z) \in B$ then
$(0, y, z) = y(0,1,0) + z(0,0,1)$
$\Rightarrow B = L(\{\alpha_1, \alpha_2\})$ when $\alpha_1 = (0,1,0)$
$\alpha_2 = (0,0,1)$
are LI
$\therefore$ The set $\{\alpha_1, \alpha_2\}$ is a basis of B.
$\therefore \dim B = 2$

Now $A \cap B = \{(0, y, 0) / y \in \mathbb{R}\}$
~~Let $(0,y,0) = y(0,1,0)$~~
$\therefore A \cap B = L(\{\beta\})$ where $\beta = (0,1,0)$ is LI.
$\therefore \{\beta\}$ is a basis of $A \cap B$.
$\therefore \dim(A \cap B) = 1$
Since $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$
$= 2 + 2 - 1$
$= 3.$

→ Find the two subspaces A and B of $V = \mathbb{R}^4(\mathbb{R})$ s.t
$\dim A = 2$, $\dim B = 3$ and $\dim(A \cap B) = 1$.

Sol^n Let $A = \{(x, y, 0, 0) / x, y \in \mathbb{R}\}$ and
$B = \{(0, y, z, t) / y, z, t \in \mathbb{R}\}$ be two subsets of $\mathbb{R}^4(\mathbb{R})$
It is easy to verify that A and B are subspaces of $V = \mathbb{R}^4(\mathbb{R})$.

Let $(x, y, 0, 0) \in A$ then

$(x, y, 0, 0) = x(1, 0, 0, 0) + y(0, 1, 0, 0)$.

$\Rightarrow A = L(\{e_1, e_2\})$.

where $e_1 = (1, 0, 0, 0)$ & $e_2 = (0, 1, 0, 0)$. are LI.

$\therefore$ The set $\{e_1, e_2\}$ is basis of A

$\therefore \dim A = 2$

Let $(0, y, z, t) \in B$ then

$(0, y, z, t) = y(0, 1, 0, 0) + z(0, 0, 1, 0) + t(0, 0, 0, 1)$.

$\Rightarrow B = L(\{\alpha_1, \alpha_2, \alpha_3\})$

where $\alpha_1 = (0, 1, 0, 0)$

$\alpha_2 = (0, 0, 1, 0)$

$\alpha_3 = (0, 0, 0, 1)$ are LI.

$\therefore$ The set $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of B.

$\therefore \dim B = 3$

Now $A \cap B = \{(0, y, 0, 0) / y \in \mathbb{R}\}$

Let $(0, y, 0, 0) \in A \cap B$ then

$(0, y, 0, 0) = y(0, 1, 0, 0)$

$A \cap B = L(\{\beta\})$

where $\beta = (0, 1, 0, 0)$ is LI.

$\therefore$ The set $\{\beta\}$ is a basis of $\mathbb{R}^4$.

$\therefore \dim(A \cap B) = 1$

$\therefore \dim(A+B) = \dim A + \dim B - \dim(A \cap B)$

$= 2 + 3 - 1$

$= 4.$

Co-ordinates :

Let $B = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of $V(F)$.

Since $B = \{\alpha_i / i = 1, 2, \ldots n\}$ spans V, the vector $\alpha \in V$ is a l.c. of the α_i's.

i.e., $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$; $a_i \in F$.

Since the α_i's are LI.

The 'n' scalars $a_1, a_2, \ldots a_n$ are completely determined by the vector α and the basis set $B = \{\alpha_i / i = 1, 2, \ldots n\}$.

and call the n-tuple $(a_1, a_2, \ldots a_n)$ the coordinate vector of α is relative to the basis $\{\alpha_i\}$ and is denoted by $[\alpha]_B$ or $[\alpha]$

i.e, $[\alpha] = (a_1, a_2, \ldots a_n)$;

problem → find the co-ordinate vector of $\alpha = (3,1,-4)$ in $\mathbb{R}^3$ ~~ofred~~ relative to the basis $f_1 = (1,1,1)$, $f_2 = (0,1,1)$ $f_3 = (0,0,1)$.

Soln α is l.c. of f_1, f_2, f_3 using unknowns x, y and z

i.e, $\alpha = xf_1 + yf_2 + zf_3$

$\Rightarrow (3,1,-4) = x(1,1,1) + y(0,1,1) + z(0,0,1)$

$= (x,x,x) + (0,y,y) + (0,0,z)$

$= (x, x+y, x+y+z)$

$\Rightarrow \boxed{x=3}$,

$x+y=1 \Rightarrow \boxed{y=-2}$

$x+y+z = -4 \Rightarrow \boxed{z=-5}$

$\therefore [\alpha] = (3,-2,-5)$

H.W → Find the co-ordinate vector of $\alpha = (3,1,-4)$ relative to the usual basis $e_1 = (1,0,0)$ $e_2 = (0,1,0)$ $e_3 = (0,0,1)$

H.W → Let V be the vector space of polynomials with degree.

$V = \{at^2 + bt + c \,/\, a, b, c \in \mathbb{R}\}$

The polynomials $e_1 = 1$, $e_2 = t-1$, and $e_3 = (t-1)^2 = t^2 - 2t + 1$ form a basis for V. Let $\alpha = 2t^2 - 5t + 6$.

find $[\alpha]$, the co-ordinate vector α relative to the basis $\{e_1, e_2, e_3\}$. (Ans: $[\alpha] = (3,-1,2)$)

→ find the coordinate vector $[u]$ relative to the basis $\{1, t, t^2, t^3\}$ of V. where $u = 2 - 3t + t^2 + 2t^3$.

Soln: u is a l.c. of $1, t, t^2, t^3$; using unknowns x, y, z, w.

i.e, $u = 1x + ty + t^2 z + t^3 w$.

$\Rightarrow 2-3t+t^2+2t^3 = x+yt+zt^2+wt^3$

$\Rightarrow x=2,\ y=-3,\ z=1,\ w=2$

$\therefore [u] = (2,-3,1,2)$

Let W be ~~the~~ the space generated by the polynomials $v_1 = t^3-2t^2+4t+1$, $v_2 = 2t^3-3t^2+9t-1$, $v_3 = t^3+6t-5$ and $v_4 = 2t^3-5t^2+7t+5$. Find a basis and dimension of W.

Sol^n Since W is spanned by polynomials of degree 3.

$\therefore W$ is a subspace of the space $V_3(\mathbb{R})$.

(the space of all real polynomials of degree ≤ 3 and the zero polynomial)

W.R.T $\{1, t, t^2, t^3\}$ is a basis for $V_3(\mathbb{R})$.

$\therefore$ The co-ordinate vectors of v_1, v_2, v_3, v_4 w.r.t the above basis are

$(1,4,-2,1)$, $(-1,9,-3,2)$, $(-5,6,0,1)$ and $(5,7,-5,2)$

Now form the matrix A whose rows are these co-ordinate vectors and reduce it to an echelon form

$$A = \begin{bmatrix} 1 & 4 & -2 & 1 \\ -1 & 9 & -3 & 2 \\ -5 & 6 & 0 & 1 \\ 5 & 7 & -5 & 2 \end{bmatrix} \underset{\substack{R_2 \to R_2+R_1 \\ R_3 \to R_3+5R_1 \\ R_4 \to R_4-5R_1}}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 26 & -10 & 6 \\ 0 & -13 & 5 & -3 \end{bmatrix}$$

$$\underset{\substack{R_3 \to R_3-2R_2 \\ R_4 \to R_4+R_2}}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in the echelon form.

The non-zero rows of the echelon form of A form a basis of the subspace W.

i.e, The vectors $(1,4,-2,1)$ and $(0,13,-5,3)$ form a basis for W.

$\therefore$ A basis for W consists of polynomials t^3-2t^2+4t+1 and $3t^3-5t^2+13t$.

$\therefore \dim W = 2$

Set-III

* Defn.

Vectorspace Homomorphism:

Let U and V be two vectorspaces over the same field F. Then the mapping $f: U \to V$ is called a homomorphism (or linear transformation) from U into V. If (i) $f(\alpha+\beta) = f(\alpha)+f(\beta)$ $\forall\ \alpha, \beta \in U$

(ii) $f(a\alpha) = a. f(\alpha)$ $\forall\ a \in F, \alpha \in U$.

— If f is onto function then V is called the homomorphic image of U.

— If 'f' is one-one onto function then 'f' is called isomorphism. Then it is said that 'U' isomorphic to V denoted by $U \cong V$.

$\longrightarrow$ Let U(F) and V(F) be two vectorspaces. Then the function $T: U \to V$ is a linear transformation of U into V iff

$T(a\alpha+b\beta) = aT(\alpha) + bT(\beta)$ $\forall\ a, b \in F$; $\alpha, \beta \in U$.

proof Now suppose that the function $T: U \to V$ is a linear transformation.

$\therefore \forall\ a, b \in F, \alpha, \beta \in U$;

$$T(a\alpha+b\beta) = T(a\alpha)+T(b\beta) \quad \text{(by defn condition (i))}$$

$$= a\,T(\alpha) + b\,T(\beta) \quad \text{(by defn condition (ii))}$$

conversely suppose that

$T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$ $\forall\ a, b \in F$; $\alpha, \beta \in U$.

Taking $a=1, b=1$ in F we get

$$T(\alpha+\beta) = T(\alpha)+T(\beta)$$

Taking $b=0$ in F we get

$$T(a\alpha) = aT(\alpha)$$

$\therefore$ T is a linear transformation.

Note(1):- The condition $T(a\alpha+b\beta) = aT(\alpha)+bT(\beta)$ completely characterises linear transformation.

Note(2): Suppose $T: U \to V$ is linear transformation. Then for any $a_i \in F$ and any $\alpha_i \in U$,

$$T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n) = a_1T(\alpha_1) + a_2T(\alpha_2) + \cdots + a_nT(\alpha_n).$$

Note(3):- If $T: U \to U$ (i.e. T transforms U into itself) then T is called a linear operator on U.

Note(4): If $T: U \to F$ (i.e. T transforms U into the field F) then T is called a linear function on U.

Zero Transformation:

Theorem: Let $U(F)$ and $V(F)$ be two vector spaces. Let the mapping $T: U \to V$ be defined by

$$T(\alpha) = \hat{0} \quad \forall\ \alpha \in U.$$

where $\hat{0}$ is the zero vector of V. Then T is a linear transformation.

Proof: $\forall$ $a, b \in F$ and $\alpha, \beta \in U$

$\implies a\alpha + b\beta \in U$ ($\because U$ is a vector space).

$\therefore$ By definition, we have

$$T(a\alpha+b\beta) = \hat{0}$$
$$= a\hat{0} + b\hat{0}$$
$$= aT(\alpha) + bT(\beta).$$

$\therefore$ T is a linear transformation.

Such a L.T. is called the zero transformation and is denoted by '$\hat{0}$'.

Theorem Let $V(F)$ be a vectorspace and the mapping $I: V \to V$ be defined by $I(\alpha) = \alpha \; \forall \; \alpha \in V$. Then, I is a linear operator from V into itself.

proof $\forall \; a, b \in F$; $\alpha, \beta \in V$

$\Rightarrow a\alpha + b\beta \in V$ ($\because V$ is a vectorspace).

By defn, we have

$$I(a\alpha + b\beta) = a\alpha + b\beta$$
$$= a\, I(\alpha) + b\, I(\beta).$$

$\therefore I$ is a linear transformation from V into ~~itself~~ itself and is called the identity operator.

*** Negative of L.T. Transformation :-**

Theorem Let $U(F)$ and $V(F)$ be two vectorspaces and $T: U \to V$ be a linear transformation. Then the mapping $(-T)$ defined by $(-T)(\alpha) = -T(\alpha)$ $\forall \; \alpha \in U$ is a linear Transformation.

proof $\forall \; a, b \in F$ and $\alpha, \beta \in U$

$\Rightarrow a\alpha + b\beta \in U$ ($\because U$ is a vectorspace)

Now by Definition,

$$(-T)(a\alpha + b\beta) = -T(a\alpha + b\beta)$$
$$= -[a\,T(\alpha) + b\,T(\beta)] \quad (\because T \text{ is L.T.})$$
$$= -a\,T(\alpha) - b\,T(\beta)$$
$$= a[-T(\alpha)] + b[-T(\beta)]$$
$$= a[(-T)(\alpha)] + b[(-T)(\beta)]$$

$\Rightarrow -T$ is L.T.

Properties of Linear transformation:

Let $T: U \to V$ be a linear transformation from the vector space $U(F)$ to the vector space $V(F)$. Then

(i) $T(\bar{0}) = \hat{0}$ where $\bar{0} \in U$ and $\hat{0} \in V$

(ii) $T(-\alpha) = -T(\alpha) \ \forall \ \alpha \in U$

(iii) $T(\alpha - \beta) = T(\alpha) - T(\beta) \ \forall \ \alpha, \beta \in U$.

Sol (i) $\alpha, \bar{0} \in U \Rightarrow T(\alpha), T(\bar{0}) \in V$

Now $T(\alpha) + T(\bar{0}) = T(\alpha + \bar{0})$ $(\because T \text{ is LT})$

$= T(\alpha)$

$= T(\alpha) + \hat{0}$ $(\because \hat{0} \in V)$

$\therefore T(\alpha) + T(\bar{0}) = T(\alpha) + \hat{0}$

By L.C.L, $T(\bar{0}) = \hat{0}$.

(ii) $T(-\alpha) = T(-1 \cdot \alpha)$

$= (-1)\, T(\alpha)$

$= -T(\alpha)$.

(iii) $T(\alpha - \beta) = T[\alpha + (-\beta)]$

$= T(\alpha) + T(-\beta)$ $(\because T \text{ is LT})$

$= T(\alpha) - T(\beta)$ (by (ii))

Determination of Linear Transformation:-

Let $U(F)$ and $V(F)$ be two vector spaces and $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a basis of U. Let $\{\delta_1, \delta_2, \dots, \delta_n\}$ be a set of n vectors in V. Then there exists a unique linear transformation $T: U \to V$ s.t

$T(\alpha_i) = \delta_i$ for $i = 1, 2, \dots, n$.

Proof

Let $U(F)$ and $V(F)$ be two vector spaces.

~~Let~~ $S_1 = \{\alpha_1, \alpha_2, \cdots \alpha_n\}$ be a basis of $U(F)$

$\therefore$ S is L.I and 'S' spans $U(F)$.

i.e $L(S_1) = U$

Let $\alpha \in U$, $\exists\, a_1, a_2, \cdots a_n \in F$ s.t

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ $(\in U)$ — (1)

(i) Existence of T:

Let $S_2 = \{\delta_1, \delta_2, \cdots \delta_n\} \subseteq V$

~~then~~ $\delta_1, \delta_2, \cdots \delta_n \in V \Rightarrow (a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n) \in V$

we define $T : U \to V$ s.t

$T(\alpha) = a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n$ — (2) where $\alpha = a_1\alpha_1 + a_2 + \cdots + a_n\alpha_n$

$\therefore$ T is a fn from U into V.

Now $\alpha_i = 0\cdot\alpha_1 + 0\alpha_2 + \cdots + 1\cdot\alpha_i + 0\alpha_{i+1} + \cdots + 0\alpha_n$.

By defn of T fn

$T(\alpha_i) = T(0\alpha_1 + 0\alpha_2 + \cdots + 1\alpha_i + 0\alpha_{i+1} + \cdots + 0\alpha_n)$

$= 0\delta_1 + 0\delta_2 + \cdots + 1\delta_i + 0\delta_{i+1} + \cdots + 0\delta_n$ (by (2)).

$= \delta_i \quad \forall i = 1, 2, 3, \cdots n.$ — (3)

(ii) To show that T is L.T:

Let $a, b \in F$ and $\alpha, \beta \in U$

$\therefore$ $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$\beta = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$.

$\therefore$ $T(\alpha) = a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n$

& $T(\beta) = b_1\delta_1 + b_2\delta_2 + \cdots + b_n\delta_n$.

$\therefore \; a\alpha + b\beta = a(a_1\alpha_1 + \cdots + a_n\alpha_n) +$

$\qquad b(b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n)$

$= (aa_1 + bb_1)\alpha_1 + \cdots + (aa_n + bb_n)\alpha_n$

$T(a\alpha + b\beta) = T\left[(aa_1+bb_1)\alpha_1 + \cdots + (aa_n+bb_n)\alpha_n\right]$

$= (aa_1+bb_1)\delta_1 + \cdots + (aa_n+bb_n)\delta_n$

(by (2))

$= a(a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n) +$

$\quad b(b_1\delta_1 + b_2\delta_2 + \cdots + b_n\delta_n)$

$= aT(\alpha) + bT(\beta).$

$\therefore$ T is L.T

(iii) To show that T is unique:-

Let $T' : U \to V$ be another L.T s.t

$T'(\alpha_i) = \delta_i$; for $i = 1, 2, 3, \ldots, n$.

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

Then $T'(\alpha) = T'(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$

$= a_1 T'(\alpha_1) + a_2 T'(\alpha_2) + \cdots + a_n T'(\alpha_n)$

($\because$ T' is L.T)

$= a_1\delta_1 + a_2\delta_2 + \cdots + a_n\delta_n$

$= T(\alpha)$.

$\therefore$ $T' = T$ and hence T is unique.

Note:- In determining the L.T the assumption that

$\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is basis of U is essential.

→ Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $S = \{\beta_1, \beta_2, \ldots \beta_n\}$ be two ordered bases of 'n' dimensional vectorspace $V(F)$. (62)

Let $\{a_1, a_2, \ldots a_n\}$ be an ordered set of 'n' scalars such that

$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$

$\beta = a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$. Then

$T(\alpha) = \beta$ where T is the linear operator on V defined by $T(\alpha_i) = \beta_i$, $i = 1, 2, 3, \ldots n$.

Sol $T(\alpha) = T(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n)$

$= a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n)$ $(\because T$ is L.T$)$

$= a_1\beta_1 + a_2\beta_2 + \cdots + a_n\beta_n$

$= \beta$.

problems

→ The mapping $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ is defined by $T(x, y, z) = (x-y, x-z)$. Show that T is a linear transformation.

Sol. Let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ be two vectors of $V_3(\mathbb{R})$.

for $a, b \in \mathbb{R}$,

$T(a\alpha + b\beta) = T[a(x_1, y_1, z_1) + b(x_2, y_2, z_2)]$

$= T[(ax_1, ay_1, az_1) + (bx_2, by_2, bz_2)]$

$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$

$= ((ax_1 + bx_2) - (ay_1 + by_2), (ax_1 + bx_2) - (az_1 + bz_2))$

$= (a(x_1 - y_1) + b(x_2 - y_2), a(x_1 - z_1) + b(x_2 - z_2))$

$$= \left(a(x_1-y_1),\ a(x_1-z_1)\right) + \left(b(x_2-y_2),\ b(x_2-z_2)\right)$$

$$= a\left(x_1-y_1,\ x_1-z_1\right) + b\left(x_2-y_2,\ x_2-z_2\right)$$

$$= a\,T(x_1,y_1,z_1) + b\,T(x_2,y_2,z_2) \quad \text{(by defn).}$$

$$= a\,T(\alpha) + b\,T(\beta).$$

$\therefore T(a\alpha + b\beta) = a\,T(\alpha) + b\,T(\beta)\ \forall\ a,b \in F;\ \alpha,\beta \in V_3(\mathbb{R})$.

$\therefore$ T is a linear transformation from $V_3(\mathbb{R})$ to $V_2(\mathbb{R})$.

→ The mapping $T: V_3(\mathbb{R}) \to V_1(\mathbb{R})$ is defined by $T(a,b,c) = a^2+b^2+c^2$. Can T be a linear transformation?

<u>Sol.</u> Let $\alpha = (a,b,c)$ and $\beta = (x,y,z)$ be two vectors of $V_3(\mathbb{R})$.

for $p,q \in \mathbb{R}$.

$$T(p\alpha + q\beta) = T\left[p(a,b,c) + q(x,y,z)\right]$$

$$= T\left[(pa,pb,pc) + (qx,qy,qz)\right]$$

$$= T\left[pa+qx,\ pb+qy,\ pc+qz\right]$$

$$= (pa+qx)^2 + (pb+qy)^2 + (pc+qz)^2 \quad \text{(by hyp)}$$

Now $p\,T(\alpha) + q\,T(\beta) = p\,T(a,b,c) + q\,T(x,y,z)$

$$= p(a^2+b^2+c^2) + q(x^2+y^2+z^2) \quad \text{(by hyp).}$$

$\therefore T(p\alpha + q\beta) \neq p\,T(\alpha) + q\,T(\beta)$.

$\therefore$ T is not a L.T from $V_3(\mathbb{R})$ to $V_1(\mathbb{R})$.

polynomials in the variable x over $\mathbb{R}$.

Let $f(x) \in V(\mathbb{R})$:- show that

(i) $D: V \to V$ defined by $D f(x) = \frac{d}{dx} f(x)$

(ii) $I: V \to V$ defined by $I f(x) = \int_0^x f(x)\,dx$

are linear Transformations.

sol Let $f(x), g(x) \in V(\mathbb{R})$ and $a, b \in \mathbb{R}$

(i) $D[a f(x) + b\, g(x)] = \frac{d}{dx}[a f(x) + b g(x)]$ (by hyp)

$= \frac{d}{dx}[a f(x)] + \frac{d}{dx}[b\, g(x)]$

$= a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x)$

$= a\, D f(x) + b\, D g(x)$

$\therefore D$ is a linear transformation and

D is called differential operator.

(ii) $I[a f(x) + b g(x)] = \int_0^x (a f(x) + b g(x))\,dx$

$= \int_0^x (a f(x))\,dx + \int_0^x (b\, g(x))\,dx$

$= a\int_0^x f(x)\,dx + b\int_0^x g(x)\,dx$.

$= a\, I f(x) + b\, I g(x)$.

$\therefore I$ is L.T. and I is called integral operator.

→ Let $P_n(\mathbb{R})$ be the vectorspace of all polynomials of degree 'n' over a field $\mathbb{R}$.

If a linear operator T on $P_n(\mathbb{R})$ is such that

$T f(x) = f(x+1)$, $f(x) \in P_n(\mathbb{R})$.

Show that $T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots + \frac{D^n}{n!}$

Let $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ $\forall a_i \in \mathbb{R}$.

Now $\left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}\right] f(x)$

$$= \left[1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}\right] (a_0 + a_1x + \cdots + a_nx^n)$$

$$= (a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + \frac{1}{1!}(0 + a_1 + 2a_2x + \cdots + na_nx^{n-1})$$

$$+ \frac{1}{2!}(0 + 0 + 2a_2 + 6a_3x + \cdots + n(n-1)a_nx^{n-2})$$

$$+ \cdots$$

$$+ \frac{1}{n!}(0 + 0 + \cdots + a_n n!)$$

$$= a_0 + a_1(x+1) + a_2(x+1)^2 + \cdots + a_n(x+1)^n$$

$$= f(x+1).$$

$$= T\,f(x). \quad (\text{by hyp.}).$$

$$\therefore T = \left(1 + \frac{D}{1!} + \frac{D^2}{2!} + \cdots + \frac{D^n}{n!}\right).$$

→ Is the mapping $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x, y, z) = (|x|, 0)$ a linear transformation?

Sol We have $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, 0).$$

Let $\alpha, \beta \in \mathbb{R}^3$ where $\alpha = (x_1, y_1, z_1)$ &

$\beta = (x_2, y_2, z_2)$

For $a, b \in \mathbb{R}$,

$$a\alpha + b\beta = a(x_1, y_1, z_1) + b(x_2, y_2, z_2)$$

$$= (ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$\therefore T(a\alpha + b\beta) = T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$

$$= (|ax_1 + bx_2|, 0).$$

And $aT(\alpha) + bT(\beta) = aT(x_1, y_1, z_1) + bT(x_2, y_2, z_2)$

$$= a(|x_1|, 0) + b(|x_2|, 0)$$

$$= (a|x_1| + b|x_2|, 0).$$

Clearly $T(a\alpha + b\beta) \neq a\,T(\alpha) + b\,T(\beta)$, (64)

Hence T is not a linear transformation.

→ Let T be a linear transformation on a vectorspace U into V (i.e $T: U \to V$ is LT). Prove that the vectors $x_1, x_2, \ldots, x_n \in U$ are L.I. if $T(x_1), T(x_2), \ldots, T(x_n)$ are L.I.

Sol. Given $T: U(F) \to V(F)$ is L.T.

and $x_1, x_2, \ldots, x_n \in U$.

Let there exist $a_1, a_2, \ldots, a_n \in F$ s.t

$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \bar{0}$ —— (1)

$(\because \bar{0} \in U)$.

$\Rightarrow T(a_1x_1 + a_2x_2 + \cdots + a_nx_n) = T(\bar{0})$

$\Rightarrow a_1T(x_1) + a_2T(x_2) + \cdots + a_nT(x_n) = \hat{0}$ $(\because T$ is LT$)$. $\hat{0} \in V(F)$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$ $(\because T(x_1), T(x_2), \ldots, T(x_n)$ are L.I.$)$

From (1) $x_1, x_2, \ldots, x_n$ are L.I.

→ Let V be a vectorspace of $n\times n$ matrices over the field F. M is a fixed matrix in V. The mapping $T: V \to V$ is defined by $T(A) = AM + MA$ where $A \in V$. Show that T is linear.

Sol. Let $a, b \in F$ and $A, B \in V$. Then

$T(A) = AM + MA$ & $T(B) = BM + MB$.

$\therefore T(aA + bB) = (aA + bB)M + M(aA + bB)$

$= a(AM + MA) + b(BM + MB)$

$= a\,T(A) + b\,T(B)$.

$\therefore T$ is a linear transformation.

→ Describe explicitly the linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(2,3)=(4,5)$ and $T(1,0)=(0,0)$.

Sol: first of all we have to show that the set $S=\{(2,3),(1,0)\}$ is a basis of $\mathbb{R}^2$.

~~for this we have~~ to show S is L.I and $L(S)=\mathbb{R}^2$.

Let $a(2,3)+b(1,0)=\bar{0}$; $a,b\in\mathbb{R}$.

$\Rightarrow (2a+b,\ 3a+0)=(0,0)$

$\Rightarrow 2a+b=0,\ 3a=0.$

$\Rightarrow a=0,\ b=0.$

$\therefore S$ is L.I

W.K.T $L(S)\subseteq\mathbb{R}^2$ —— (1).

Let $(x,y)\in\mathbb{R}^2$ then $(x,y)=a(2,3)+b(1,0)$

$\Rightarrow (x,y)=(2a+b,\ 3a+0)$

$\Rightarrow 2a+b=x;\ 3a=y$

$\Rightarrow 2\left(\frac{y}{3}\right)+b=x;\ \boxed{a=\frac{y}{3}}$

$\Rightarrow \boxed{b=x-\frac{2y}{3}}$

$\therefore (x,y)=\frac{y}{3}(2,3)+\left(x-\frac{2y}{3}\right)(1,0)$

$\in L(S).$

$\therefore (x,y)\in L(S)$

$\therefore \mathbb{R}^2\subseteq L(S)$ —— (2)

from (1) & (2), we have $L(S)=\mathbb{R}^2$

$\therefore S$ is a basis of $\mathbb{R}^2$.

Now let $\therefore S=\{(2,3),(1,0)\}$ is a basis of $\mathbb{R}^2$ and

$S'=\{(4,5),(0,0)\}$ is a set of two vectors in $\mathbb{R}^2$.

$\therefore \exists$ a unique linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t

$$T(x,y) = T\left[\frac{y}{3}(2,3) + \left(x - \frac{2y}{3}\right)(1,0)\right]$$

$$= \frac{y}{3}T(2,3) + \left(x - \frac{2y}{3}\right)T(1,0)$$

$$= \frac{y}{3}(4,5) + \left(x - \frac{2y}{3}\right)(0,0)$$

$$= \left(\frac{4y}{3}, \frac{5y}{3}\right)$$ which is the reqd transformation.

HW → Find $T(x,y,z)$ where $T: \mathbb{R}^3 \to \mathbb{R}$ is defined by $T(1,1,1) = 3$, $T(0,1,-2) = 1$, $T(0,0,1) = -2$.

HW → Find a linear transformation

(i) $T: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$

(ii) $T: V_2 \to V_2$ s.t $T(1,2) = (3,0)$ and $T(2,1) = (1,2)$

(iii) $T: V_3 \to V_3$ s.t $T(0,1,2) = (3,1,2)$ and $T(1,1,1) = (2,2,2)$

(iv) $T: V_2(\mathbb{R}) \to V_3(\mathbb{R})$ s.t $T(1,1) = (3,-1,5)$ and $T(0,1) = (2,1,-1)$

* Sum of Linear Transformations

Defn → Let T_1 and T_2 be two linear transformations from $U(F)$ into $V(F)$. Then their sum $T_1 + T_2$ is defined by $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall \alpha \in U$.

Theorem → Let $U(F)$ and $V(F)$ be two vector spaces. Let T_1 and T_2 be two linear transformations from U into V. Then the mapping $T_1 + T_2$ defined by $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha)\ \forall \alpha \in U$ is a linear transformation.

→ Show that $B = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of $\mathbb{R}^3$. Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a L.T such that $T(1,0,0) = (1,0,0)$, $T(1,1,0) = (1,1,1)$ and $T(1,1,1) = (1,1,0)$. Find $T(x,y,z)$.

Proof: Given the $T_1 : U \to V$ and $T_2 : U \to V$ are linear transformation and $(T_1+T_2)(\alpha) = T_1(\alpha) + T_2(\alpha) \;\forall\, \alpha \in U$, $T_1(\alpha), T_2(\alpha) \in V$

Since $T_1(\alpha), T_2(\alpha) \in V$

$\Longrightarrow T_1(\alpha) + T_2(\alpha) \in V$.

Hence $(T_1+T_2) : U \to V$

Let $a, b \in F$ and $\alpha, \beta \in U$. Then

$$(T_1+T_2)(a\alpha + b\beta) = T_1(a\alpha+b\beta) + T_2(a\alpha+b\beta) \quad \text{(by hyp)}$$

$$= \left(aT_1(\alpha) + bT_1(\beta)\right) + \left(aT_2(\alpha) + bT_2(\beta)\right) \quad (\because T_1 \,\&\, T_2 \text{ are L.T})$$

$$= a\left(T_1(\alpha) + T_2(\alpha)\right) + b\left(T_1(\beta) + T_2(\beta)\right).$$

$$= a(T_1+T_2)(\alpha) + b(T_1+T_2)(\beta) \quad \text{(by hyp)}$$

$\therefore T_1 + T_2$ is a L.T from U into V.

* **Scalar multiplication of a L.T:-**

$\rightarrow$ Let $T : U(F) \to V(F)$ be a linear transformation and $a \in F$. Then the function (aT) defined by
$$(aT)(\alpha) = aT(\alpha) \;\forall\, \alpha \in U.$$ is a linear transformation.

Proof: Given the $T : U(F) \to V(F)$ and $(aT)(\alpha) = aT(\alpha) \;\forall\, a \in F, \alpha \in U$

Now $T(\alpha) \in V \Rightarrow a\,T(\alpha) \in V$

$\therefore (aT) : U \to V$.

For $c, d \in F$ and $\alpha, \beta \in U$

$$\Longrightarrow (aT)(c\alpha + d\beta) = aT(c\alpha + d\beta) \quad \text{(by hyp)}.$$

$= a\left[c\,T(\alpha) + d\,T(\beta)\right]$ $(\because T$ is L.T.$)$

$= ac\,T(\alpha) + ad\,T(\beta).$

$= c\,(aT)(\alpha) + d\,(aT)(\beta).$

Hence (aT) is a L.T. from U into V.

problems

→ Let $T: V_3(R) \to V_2(R)$ and $H: V_3(R) \to V_2(R)$ be the two linear transformations defined by $T(x,y,z) = (x-y,\ y+z)$ and $H(x,y,z) = (2x,\ y-z)$.

Find (i) $H+T$ (ii) aH.

Sol (i) $(H+T)(x,y,z) = H(x,y,z) + T(x,y,z)$

$= (2x,\ y-z) + (x-y,\ y+z)$

$= (3x-y,\ 2y)$

(ii) $(aH)(x,y,z) = a\,H(x,y,z)$

$= a\,(2x,\ y-z)$

$= (2ax,\ ay-az)$

→ Let $G: V_3 \to V_3$ and $H: V_3 \to V_3$ be two linear operators defined by

$G(e_1) = e_1+e_2,\quad G(e_2) = e_3,\quad G(e_3) = e_2-e_3$

and $H(e_1) = e_3,\quad H(e_2) = 2e_2-e_3,\quad H(e_3) = 0.$

where $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(R)$.

Find (i) $G+H$ (ii) αG.

Sol Let $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $V_3(R)$.

Sol^n:-

$e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$.

Now $G(e_1) = e_1 + e_2$

$\Rightarrow G(1,0,0) = (1,0,0) + (0,1,0)$

$= (1,1,0)$

$\therefore \boxed{G(1,0,0) = (1,1,0)}$

$G(e_2) = e_3$

$\Rightarrow \boxed{G(0,1,0) = (0,0,1)}$

$G(e_3) = e_2 - e_3$

$\Rightarrow G(0,0,1) = (0,1,0) - (0,0,1)$

$\boxed{G(0,0,1) = (0,1,-1)}$

Again $H(e_1) = e_3$

$\Rightarrow \boxed{H(1,0,0) = (0,0,1)}$

$H(e_2) = 2e_2 - e_3$

$\Rightarrow \boxed{H(0,1,0) = (0,2,-1)}$

$H(e_3) = 0$

$\Rightarrow \boxed{H(0,0,1) = (0,0,0)}$

(i) $(G+H)(e_1) = G(e_1) + H(e_1)$

$\Rightarrow (G+H)(1,0,0) = (1,1,0) + (0,0,1)$

$= (1,1,1)$

$(G+H)(e_2) = G(e_2) + H(e_2) \Rightarrow (G+H)(0,1,0) = (0,2,0)$

$(G+H)(e_3) = G(e_3) + H(e_3) \Rightarrow (G+H)(0,0,1) = (0,1,-1)$

(ii) $(2G)(e_1) = 2G(e_1) = 2e_1 + 2e_2$

$(2G)(e_2) = 2G(e_2) = 2e_3$

$(2G)(e_3) = 2G(e_3) = 2e_2 - 2e_3$ etc.

* Product of Linear Transformation :-

→ Let $U(F)$, $V(F)$ and $W(F)$ are three vector spaces and $T: V \to W$ and $H: U \to V$ are two linear transformations. Then the composite function TH (called the product of linear transformations) defined by

$$(TH)(\alpha) = T[H(\alpha)] \quad \forall \alpha \in U.$$

is a linear transformation from U into W.

Note: The range of H is the domain of T.

→ Let H, H' be two linear transformations from $U(F)$ to $V(F)$. Let T, T' be the linear transformations from $V(F)$ to $W(F)$ and $a \in F$. Then

(i) $T(H+H') = TH + TH'$

(ii) $(T+T')H = TH + T'H$

(iii) $a(TH) = (aT)H = T(aH)$

→ Algebra of Linear Operators :

→ Let A, B, C be linear operators on a vector space $V(F)$. Also let O be the zero-operator and I the Identity operator on V. Then

(i) $AO = OA = O$

(ii) $AI = IA = A$

(iii) $A(B+C) = AB + AC$

(iv) $(A+B)C = AC + BC$

(v) $A(BC) \overset{!}{=} (AB)C$

→ Let $T: R^3 \to R^2$ and $H: R^3 \to R^2$ be defined by $T(x, y, z) = (3x, y+z)$ and $H(x, y, z) = (2x - z, y)$

Compute (i) $T+H$ (ii) $4T - 5H$ (iii) TH (iv) HT.

Sol Since T and H map $\mathbb{R}^3 \to \mathbb{R}^2$, the linear transformations $T+H$ and $4T-5H$ are defined.

(i) $(T+H)(x,y,z) = T(x,y,z) + H(x,y,z)$

$= (3x, y+z) + (2x-z, y)$

$= (5x-z, 2y+z)$.

(ii) $(4T-5H)(x,y,z) = 4T(x,y,z) - 5H(x,y,z)$

$= 4(3x, y+z) - 5(2x-z, y)$

$= (2x+5z, -y+4z)$.

(iii) and (iv) both TH and HT are not defined because the range of T is not equal to the domain of H and vice versa.

→ Let $T_1 : \mathbb{R}^3 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are two linear transformations defined by

$T_1(x,y,z) = (3x, 4y-z)$

$T_2(x,y) = (-x, y)$. Compute T_1T_2 and T_2T_1.

Sol (i) Since the range of T_2 i.e $\mathbb{R}^2$ is not equal to the domain of T_1 (i.e $\mathbb{R}^3$)

$\therefore$ T_1T_2 is not defined.

(ii) the range T_1 i.e $\mathbb{R}^3$ is equal to the domain of T_2

$\therefore$ T_2T_1 is defined.

$\therefore (T_2T_1)(x,y,z) = T_2[T_1(x,y,z)]$

$= T_2[3x, 3y-z]$

$= (-3x, 3y-z)$

Let $P(\mathbb{R})$ be the vectorspace of all polynomials over $\mathbb{R}$ & D, T be two linear operators on P defined by $D[f(x)] = \frac{df}{dx}$ and

$T[f(x)] = x f(x) \;\forall\; f(x) \in P(\mathbb{R})$

Show that (i) $TD \neq DT$ (ii) $(TD)^2 = T^2D^2 + TD$

Sol (i) $(T.D)\, f(x) = T[D f(x)]$

$= T\left[\frac{df}{dx}\right]$ (by hyp)

$= x f'(x)$

and $(DT)\, f(x) = D[T f(x)]$

$= D[x f(x)]$ (by hyp)

$= \frac{d}{dx}(x f(x))$

$= x f'(x) + f(x)$

Clearly $TD \neq DT$

Also $(DT)\, f(x) - (TD) f(x) = f(x)$

$\Rightarrow (DT - TD) f(x) = I f(x)$

$\Rightarrow (DT - TD) = I$

(ii) $(TD)^2 f(x) = [(TD)(TD)] f(x)$

$= (TD)[(TD) f(x)]$

$= (TD)[x f'(x)]$

$= T[D(x f'(x))]$

$= T\left[\frac{d}{dx}(x f'(x))\right]$

$= T[(x f''(x) + f'(x))]$

$= T[x f''(x) + f'(x)]$

$= x[x f''(x) + f'(x)]$

$= x^2 f''(x) + x f'(x)$

Now $(T^2D^2)f(x)$

$$= T^2 D\,[D\,f(x)]$$

$$= T^2 D\left[\frac{df}{dx}\right]$$

$$= T^2\left[\frac{d^2f}{dx^2}\right]$$

$$= T\left(T\left[\frac{d^2f}{dx^2}\right]\right)$$

$$= T\left(x\frac{d^2f}{dx^2}\right)$$

$$= x\left(x\frac{d^2f}{dx^2}\right)$$

$$= x^2\frac{d^2f}{dx^2}.$$

$$\therefore (T^2D^2+TD)(f(x)) = (T^2D^2)(f(x)) + (TD)f(x)$$

$$= x^2\frac{d^2f}{dx^2} + x\frac{df}{dx}$$

$$\therefore (TD)^2(f(x)) = (T^2D^2+TD)\,f(x) \quad \forall f(x)\in P$$

$$(TD)^2 = T^2D^2 + TD.$$

HW $\rightarrow$ Let P be the polynomial space in one indeterminate x with real co-efficients. Let $D: P\to P$ and $S: P\to P$ be two linear operators defined by

$$Df(x) = \frac{df}{dx} \quad\text{and}\quad S\,f(x) = \int_0^x f(x)\,dx \quad \forall f(x)\in P.$$

Show that $DS = I$ and $SD \neq I$ where I is the identity transformation.

HW → Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ and $H: \mathbb{R}^2 \to \mathbb{R}^3$ be two linear transformations defined by

$T(x, y, z) = (x - 3y - 2z, y - 4z)$.

and $H(x, y) = (2x, 4x - y, 2x + 3y)$

Find HT and TH. Is product commutative

HW → Define on $\mathbb{R}^2$ linear operators H and T as follows $H(x, y) = (0, x)$ and $T(x, y) = (x, 0)$

and show that

$T.H = 0$, $HT \neq TH$ and $T^2 = T$.

*** Transformations as vectors ***

→ Let $L(U, V)$ be the set of ~~all~~ linear transformations from a vector space $U(F)$ into a vectorspace $V(F)$. Then $L(U, V)$ be a vector space relative to the operations of vector addition and scalar multiplication defined as

(i) $(T+H)(\alpha) = T(\alpha) + H(\alpha)$. $\alpha \in U$, $a \in F$ and

(ii) $(aT)(\alpha) = aT(\alpha)$ $\forall$ $T, H \in L(U, V)$.

∴ The set $L(U, V)$ is also denoted by $Hom(U, V)$.

proof Let $L(U, V) = \{T : U \to V / T \text{ is a L.T}\}$

Given that $T : U \to V$ and $H : U \to V$ are L.T. en $L(U, V)$

and $(T+H)(\alpha) = T(\alpha) + H(\alpha)$ $\forall \alpha \in U$.

$T(\alpha), H(\alpha) \in V$.

Since $T(\alpha)$, $H(\alpha) \in V$

$\Longrightarrow T(\alpha) + H(\alpha) \in V$.

∴ $(T+H) : U \to V$.

Let $a, b \in F$ and $\alpha, \beta \in U$ then

$(T+H)(a\alpha+b\beta) = T(a\alpha+b\beta)+H(a\alpha+b\beta)$

$= (a\,T(\alpha)+b\,T(\beta)) + (a\,H(\alpha)+b\,H(\beta))$

($\because$ T & H are L.T.)

$= a(T+H)(\alpha) + b(T+H)(\beta)$.

$\therefore$ $T+H$ is a L.T. from U into V.

$\therefore$ $T+H \in L(U,V)$.

$\therefore$ Internal composition is satisfied by L(U,V)

Given that $T: U \to V$ is L.T. in $L(U,V)$.

and $(aT)(\alpha) = a\,T(\alpha)$ $\forall$ $a \in F$, $\alpha \in V$

Now $T(\alpha) \in V \Rightarrow a\,T(\alpha) \in V$

$\therefore$ $(aT): U \to V$.

for $c, d \in F$ and $\alpha, \beta \in U$

$\Rightarrow (aT)(c\alpha+d\beta) = aT(c\alpha+d\beta)$

(by hyp)

$= a[c\,T(\alpha)+d\,T(\beta)]$

$= a\,c\,T(\alpha)+a\,d\,T(\beta)$

$= c\,(aT)(\alpha)+d\,(aT)(\beta)$

$\therefore$ (aT) is a L.T. from U into V.

$\therefore$ $aT \in L(U,V)$.

$\therefore$ External composition is satisfied in L(U,V) over the field F.

(I) (i) $\forall$ $T, H \in L(U,V) \Rightarrow T+H \in L(U,V)$

$\therefore$ closure prop is satisfied.

(ii) $\forall$ $T, H, G \in L(U,V)$.

$[(T+H)+G](\alpha) = (T+H)(\alpha) + G(\alpha)$

$= [T(\alpha)+H(\alpha)] + G(\alpha)$

$= T(\alpha) + [H(\alpha) + G(\alpha)]$ $\because$ + is associative

$= T(\alpha) + [H + G](\alpha)$

$= [T + (H + G)](\alpha)$

$\therefore (T + H) + G = T + (H + G)$

$\therefore$ Asso. prop. is satisfied in $L(V, V)$

(iii) Let 'O' be the zero transformation from V into V

i.e. $O(\alpha) = \hat{0}$ $\forall \alpha \in V$, $\hat{0} \in V$

Now $(O + T)(\alpha) = O(\alpha) + T(\alpha)$

$= \hat{0} + T(\alpha)$

$= T(\alpha)$ ($\because \hat{0}$ is additive identity in V)

$\therefore O + T = T$

$\text{lly } T + O = T$

$\therefore \forall T \in L(V, V)$ $\exists O \in L(V, V)$ s.t.

$O + T = T = T + O$

Hence 'O' is the additive identity for $L(V, V)$.

(iv) For $T \in L(V, V)$,

let us define $(-T)$ as $(-T)(\alpha) = -T(\alpha)$ $\forall \alpha \in V$.

Then $(-T) \in L(V, V)$.

Now $(-T + T)(\alpha) = (-T)(\alpha) + T(\alpha)$

$= -T(\alpha) + T(\alpha)$

$= \hat{0}$ ($\because \hat{0} \in V$).

$= O(\alpha)$

$\therefore (-T) + T = O$ $\forall T \in L(V, V)$.

$\text{lly } T + (-T) = O$ $\forall T \in L(V, V)$.

$\therefore \forall T \in L(V, V), \exists -T \in L(V, V)$ s.t.

$(-T) + T = O = T + (-T)$

~~Here~~ $-T$ is additive inverse of T for $L(V,V)$.

(v) $(T+H)(\alpha) = T(\alpha) + H(\alpha)$

$= H(\alpha) + T(\alpha)$ ($\because$ addition in V is commutative)

$= (H+T)(\alpha)$

$\therefore T+H = H+T$.

$\therefore$ commutative prop is satisfied for $L(V,V)$

$\therefore (L(V,V), +)$ is an abelian grp.

(II) $\forall\ a,b \in F,\ T,H \in L(V,V)$;

$\Rightarrow$ (i) $[a(T+H)](\alpha) = a\,(T+H)(\alpha)$ (by hyp (iii))

$= a\,[T(\alpha) + H(\alpha)]$ (by hyp (ii))

$= a\,T(\alpha) + a\,H(\alpha)$

$= (aT)(\alpha) + (aH)(\alpha)$ (by hyp (iii))

$= (aT + aH)(\alpha)$ (by hyp (i))

$\therefore a(T+H) = aT + aH$.

(ii) $[(a+b)T](\alpha) = (a+b)\,T(\alpha)$ (by hyp (ii))

$= a\,T(\alpha) + b\,T(\alpha)$

$= (aT)(\alpha) + (bT)(\alpha)$ (by hyp (iii))

$= (aT + bT)(\alpha)$.

$\therefore (a+b)T = aT + bT$

(iii) $[(ab)T](\alpha) = ab\,(T(\alpha))$ (by hyp)

$= a\,(b\,T(\alpha))$

$= a[(bT)(\alpha)] = [a(bT)](\alpha)$

$\therefore (ab)T = a(bT)$

(iv) $(1 \cdot T)(\alpha) = 1 \cdot T(\alpha)$

$= T(\alpha)$; (multiplication in V is identity). $1 \in F$

$\therefore 1 \cdot T = T$

$\therefore L(U, V)$ is a vectorspace over the field F.

→ $L(U,V)$ be the vectorspace of all linear transformations from $U(F)$ to $V(F)$. So that $\dim U = n$ and $\dim V = m$.

Then $\dim L(U,V) = mn$.

<u>proof</u> Given that $L(U,V)$ is the vectorspace of all linear transformations from $U(F)$ to $V(F)$

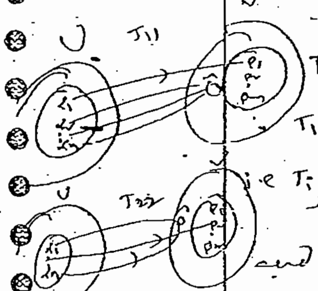
i.e $L(U,V) = \{T : U \to V / T$ is a linear transformation$\}$.

Since $\dim U = n$ and $\dim V = m$

Let $B_1 = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and

$B_2 = \{\beta_1, \beta_2, \ldots \beta_m\}$ be the ordered bases of U and V respectively.

$\therefore$ there exists uniquely a linear transformation T_{ij} from U to V such that



$T_{11}(\alpha_1) = \beta_1$, $T_{11}(\alpha_2) = \hat{0}, \ldots T_{11}(\alpha_n) = \hat{0}$ where $\beta_1, \hat{0} \in V$

i.e $T_{ij}(\alpha_i) = \beta_j$, $i = 1, 2, \ldots n$
$j = 1, 2, \ldots m$.

and $T_{ij}(\alpha_k) = \hat{0}$ $k \neq i$

Thus there are "mn" T_{ij}'s $\in L(U,V)$.

We shall show that $S = \{T_{ij}\}$ of mn elts is a basis for $L(U,V)$.

(i) <u>To prove S is L.I</u>:

Let a_{ij}'s $\in F$, let us suppose that $\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij} = \hat{0}$ $(\because \hat{0} \in L(U,V))$

for $\alpha_k \in V$, $k = 1, 2, 3, \ldots, n$ we get

$$\left[\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}\right](\alpha_k) = O(\alpha_k)$$

$$\Rightarrow \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} T_{ij}(\alpha_k) = \hat{0} \quad (\because \hat{0} \in V)$$

$$\Rightarrow \sum_{j=1}^{m} a_{kj} T_{kj}(\alpha_k) = \hat{0}. \quad \text{where } 1 \le k \le n.$$

$\left(\because \sum_{j=1}^{m} (a_{1j} T_{1j}(\alpha_k) + a_{2j} T_{2j}(\alpha_k) + \cdots + a_{nj} T_{nj}(\alpha_k)\right)$

$\Rightarrow \sum_{j=1}^{m} (a_{1j} T_{1j}(\alpha_1) + a_{2j} T_{2j}(\alpha_2) + \cdots + a_{nj} \ldots$

$\Rightarrow \sum_{j=1}^{m} (a_{1j} \beta_j) = 0$ if $k = 1$

$\text{lly } \sum_{j=1}^{m} (a_{2j} \beta_j) = \hat{0}$ if $k = 2$ and so on

$$\Rightarrow a_{k1} T_{k1}(\alpha_k) + a_{k2} T_{k2}(\alpha_k) + \cdots + a_{km} T_{km}(\alpha_k) = \hat{0}$$

$$\Rightarrow a_{k1}\beta_1 + a_{k2}\beta_2 + \cdots + a_{km}\beta_m = \hat{0}.$$

$$\Rightarrow a_{k1} = a_{k2} = \cdots = a_{km} = 0 \quad (\because B_2 \text{ is a basis of } V)$$

$\therefore S = \{T_{ij}\}$ is L.I. set.

(ii) To show that $L(S) = L(U, V)$.

Let $T \in L(U, V)$ then the vector $T(\alpha_i) \in V$

It can be expressed as L.C of elts of B_2.

i.e $T(\alpha_1) = b_{11}\beta_1 + b_{21}\beta_2 + \cdots + b_{m1}\beta_m$.

In general for $i = 1, 2, \ldots, n$

$$T(\alpha_i) = b_{1i}\beta_1 + b_{2i}\beta_2 + \cdots + b_{mi}\beta_m \quad \text{--- (1)}$$

Consider the linear transformation,

$$H = \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} T_{ij}$$

Clearly H is a linear combination of $S = \{T_{ij}\}$

$\therefore H \in L(U, V)$.

Let $\alpha_k \in U$ for $k = 1, \ldots, n$.

Since $T_{ij}(\alpha_k) = \hat{0}$ for $k \ne i$ & $T_{kj}(\alpha_k) = \beta_j$

we have
$$H(\alpha_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij} T_{ij}(\alpha_k)$$
$$= \sum_{j=1}^{m} b_{kj} T_{kj}(\alpha_k)$$

$$= \sum_{j=1}^{m} b_{kj} \beta_j$$

i.e $H(\alpha_k) = b_{k1}\beta_1 + b_{k2}\beta_2 + \cdots + b_{km}\beta_m$

$= T(\alpha_k)$ (by ①)

$\therefore H(\alpha_k) = T(\alpha_k)$ for each k

$\Rightarrow H = T$

Thus T is a linear combination of elts of S

i.e $L(S) = L(V, V)$

$\therefore S$ is a basis set of $L(V, V)$.

$\therefore \dim L(V,V) = mn$

problems

→ find the dimension of $L(\mathbb{R}^3, \mathbb{R}^2)$

since $\dim \mathbb{R}^3 = 3$ and

$\dim \mathbb{R}^2 = 2$

$\therefore \dim(L(\mathbb{R}^3, \mathbb{R}^2)) = 6.$

→ find the dimension of $L(C^3, \mathbb{R}^2)$.

Sol: since C^3 is a vector space over 'C'

and $\mathbb{R}^2$ is a vector space over $\mathbb{R}$

Hence i.e $\dim(L(C^3, \mathbb{R}^2))$ does not exist.

→ Let $V = C^3$ be a vector space over $\mathbb{R}$.

find the dimension of $L(V, \mathbb{R}^2)$

(i.e dim of $Hom(V, \mathbb{R}^2)$)..

Sol: Since $V = C^3$ is a vectorspace over $\mathbb{R}$

i.e let $V = \{(a_1+ib_1, a_2+ib_2, a_3+ib_3) / a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}\}$

$\therefore \dim V = 6$

and obviously $\dim \mathbb{R}^2 = 2$

$\therefore \dim(Hom(V, V)) = 6 \times 2 = 12$

from this topic (72)

* Range and Nullspace of a Linear Transformation:

Imp. **Def** Let $U(F)$ and $V(F)$ be two vectorspaces and let $T: U \to V$ be a linear transformation. The range of T is defined to be the set

$$\text{Range}(T) = R(T) = \{T(\alpha) / \alpha \in U\}.$$

Obviously the range of T is a subset of V. i.e. $R(T) \subseteq V$.

→ Let $U(F)$ and $V(F)$ be two vectorspaces. Let $T: U(F) \to V(F)$ be a linear transformation. Then the range set $R(T)$ is a subspace of $V(F)$.

Let $R(T) = \{T(\alpha) / \alpha \in U\}$

proof — for $\bar{0} \in U \Rightarrow T(\bar{0}) = \hat{0} \in R(T)$

$\therefore R(T)$ is non-empty set and $R(T) \subseteq V$.

Let $\alpha_1, \alpha_2 \in U$ and $\beta_1, \beta_2 \in R(T)$ be s.t $T(\alpha_1) = \beta_1$ and $T(\alpha_2) = \beta_2$.

— for $a, b \in F$, $a\alpha_1 + b\alpha_2 \in U$ ($\because U$ is v.s).

$$\Rightarrow T(a\alpha_1 + b\alpha_2) \in R(T).$$

But $T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2)$ ($\because T$ is LT)

$= a\beta_1 + b\beta_2$

$\in R(T)$.

$\therefore a, b \in F$ and $\beta_1, \beta_2 \in R(T)$

$\Rightarrow a\beta_1 + b\beta_2 \in R(T)$.

$\therefore R(T)$ is subspace of $V(F)$.

$R(T)$ is called the range space.

Nullspace or Kernal:

Let $V(F)$ and $V(F)$ be two vectorspaces and $T: U \to V$ be a linear transformation. The nullspace denoted by $N(T)$ is the set of all vectors $\alpha \in U$ s.t $T(\alpha) = \hat{0}$ (zero vector of V).

The nullspace of $N(T)$ is also called the kernel of T.

i.e $N(T) = \{\alpha \in U / T(\alpha) = \hat{0} \in V\}$

Obviously the nullspace $N(T) \subseteq U$.

→ Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ is a linear transformation. Then nullspace $N(T)$ is a subspace of $U(F)$.

proof Let $N(T) = \{\alpha \in U / T(\alpha) = \hat{0} \in V\}$

$\because T(\bar{0}) = \hat{0} \Rightarrow \bar{0} \in N(T)$ $\left(\because \bar{0} \in U, \hat{0} \in V\right)$.

$\therefore N(T)$ is a non-empty subset of U.

Now $\alpha, \beta \in N(T) \Rightarrow T(\alpha) = \hat{0}$, $T(\beta) = \hat{0}$.

For $a, b \in F$, $T(a\alpha + b\beta) = a \cdot T(\alpha) + b\, T(\beta)$ $(\because T$ is LT$)$

$= a \cdot \hat{0} + b \cdot \hat{0}$

$= \hat{0}$.

$\therefore T(a\alpha + b\beta) = \hat{0}$.

By definition $a\alpha + b\beta \in N(T)$.

$\therefore \forall a, b \in F$ and $\alpha, \beta \in N(T) \Rightarrow a\alpha + b\beta \in N(T)$.

$\therefore$ nullspace $N(T)$ is a subspace of $U(F)$.

→ Let $T: U(F) \to V(F)$ be a linear transformation. If U is finite dimensional then the range space $R(T)$ is a finite dimensional subspace of $V(F)$.

proof

Let $S=\{\alpha_1,\alpha_2,\cdots,\alpha_n\}$ be the basis set of $U(F)$.

Let $\beta\in R(T)$

Then $\exists$ $\alpha\in U$ such that $T(\alpha)=\beta$.

$\therefore$ $\alpha=a_1\alpha_1+a_2\alpha_2+\cdots\cdots+a_n\alpha_n$ for a_i's $\in F$.

$\Rightarrow T(\alpha)=T(a_1\alpha_1+a_2\alpha_2+\cdots-+a_n\alpha_n)$

$\Rightarrow \beta = a_1T(\alpha_1)+a_2T(\alpha_2)+\cdots+a_nT(\alpha_n)$

But $S'=\{T(\alpha_1),T(\alpha_2),\ldots T(\alpha_n)\}\subset R(T) \Rightarrow L(S')\subseteq R(T)$ — (1)

Now $\beta\in R(T)$ and $\beta=$ l.c. of elts of S'

$\Rightarrow \beta\in L(S') \Rightarrow R(T)\subseteq L(S')$ — (2)

from (1) & (2), we have $R(T)=L(S')$

~~Thus $R(T)$ is spanned~~ by a finite set S'.

$\therefore$ $R(T)$ is finite dimensional subspace of $V(F)$.

* Dimension of Range and Kernel:

Let $T:U(F)\longrightarrow V(F)$ be a linear transformation where U is finite dimensional vector space.

Rank: Then the rank of T denoted by $\rho(T)$ is the dimension of range space $R(T)$.

i.e., $\rho(T)=\dim R(T)$.

Nullity: The nullity of T denoted by $\nu(T)$ is the dimension of null space $N(T)$.

$\therefore$ $\nu(T)=\dim N(T)$.

Theorem

Let $U(F)$ and $V(F)$ be two vector spaces and $T:U\to V$ be a linear transformation. Let U be finite dimensional then $\rho(T)+\nu(T)=\dim U$.

i.e., rank $(T)+$ nullity$(T)=\dim U$.

Proof: The null space $N(T)$ is a subspace of finite dimensional space $U(F)$.

$\Rightarrow N(T)$ is finite dimensional.

Let $S = \{\alpha_1, \alpha_2, \cdots \alpha_k\}$ be a basis of $N(T)$.

$\therefore \dim N(T) = \nu(T) = k$.

$\therefore T(\alpha_1) = \hat{0},\ T(\alpha_2) = \hat{0} \cdots\cdots T(\alpha_k) = \hat{0}$. — (1)

As S is L.I. it can be extended to form a basis of $U(F)$.

Let $S_1 = \{\alpha_1, \alpha_2 \cdots \alpha_k, \theta_1, \theta_2, \cdots \theta_m\}$ be the extended basis of $U(F)$.

$\therefore \dim U = k+m$.

Now we show that the set of images of additional vectors $S_2 = \{T(\theta_1), T(\theta_2), \cdots T(\theta_m)\}$ is a basis of $R(T)$.

Clearly $S_2 \subseteq R(T)$.

(i) To prove S_2 is L.I.

Let $a_1, a_2, \cdots a_m \in F$ such that

$a_1 T(\theta_1) + a_2 T(\theta_2) + \cdots\cdots + a_m T(\theta_m) = \hat{0}$.

$\Rightarrow T(a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m) = \hat{0}$. ($\because T$ is LT)

$\Rightarrow a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m \in N(T)$.

But each vector in $N(T)$ is a l.c. of elts of basis 'S'.

$\therefore$ for some $b_1, b_2, \cdots b_k \in F$,

Let $a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_k\alpha_k$

$\Rightarrow a_1\theta_1 + a_2\theta_2 + \cdots + a_m\theta_m - b_1\alpha_1 - b_2\alpha_2 - \cdots - b_k\alpha_k = 0$

$\Rightarrow a_1 = 0,\ a_2 = 0, \cdots a_m = 0,\ b_1 = 0, b_2 = 0 \cdots b_k = 0$

($\because S_1$ is L.I.)

(ii) To prove $L(S_2) = R(T)$

Let $\beta \in$ range space $R(T)$, then $\exists\ \alpha \in U$ s.t.

$$T(\alpha) = \beta.$$

Now $\alpha \in U \Rightarrow$ there exist $c_i', d_i' \in F$ such that

$$\alpha = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \cdots + d_m\theta_m$$

$$\Rightarrow T(\alpha) = T(c_1\alpha_1 + c_2\alpha_2 + \cdots + c_k\alpha_k + d_1\theta_1 + d_2\theta_2 + \cdots + d_m\theta_m)$$

$$= c_1 T(\alpha_1) + c_2 T(\alpha_2) + \cdots + c_k T(\alpha_k) + d_1 T(\theta_1) + d_2 T(\theta_2) + \cdots + d_m T(\theta_m)$$

$$\Rightarrow \beta = d_1 T(\theta_1) + d_2 T(\theta_2) + \cdots + d_m T(\theta_m) \quad (\because \text{by } (1))$$

$$\Rightarrow \beta \in L(S_2).$$

$\therefore S_2$ is a basis of $R(T)$.

and $\dim R(T) = m$

$\therefore \dim R(T) + \dim N(T) = m + k = \dim U.$

i.e., $\rho(T) + \nu(T) = \dim U.$

$f(x+y) = f(x) + f(y)$

$\frac{d}{dx} f(x+y) = f'(x) + f'(y) \cdot \frac{dy}{dx}$

$f'(x+y) \cdot (1 + \frac{dy}{dx})$

$= f'(x) + f'(y) \cdot \frac{dy}{dx}$

$\frac{dy}{dx}(f'(y) - f'(x+y))$

$= f'(x) - f'(x+y)$

problems

→ If $T: V_4(\mathbb{R}) \to V_3(\mathbb{R})$ is a linear transformation defined by $T(a,b,c,d) = (a-b+c+d,\ a+2c-d,\ a+b+3c-3d)$ for $a,b,c,d \in \mathbb{R}$ then verify $\rho(T) + \nu(T) = \dim V_4(\mathbb{R})$.

Sol. Let $S = \{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ be the standard basis set of $V_4(\mathbb{R})$

$\therefore$ the transformation T on S will be

$T(1,0,0,0) = (1,1,1)$, $T(0,1,0,0) = (-1,0,1)$

$T(0,0,1,0) = (1,2,3)$, $T(0,0,0,1) = (1,-1,-3)$

Let $S_1 = \{(1,1,1), (-1,0,1), (1,2,3), (1,-1,-3)\}$

$\therefore S_1 \subseteq R(T)$.

Now we verify whether S_1 is L.I or not.

If not, we find least L.I set by forming the matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2 \quad R_4 \to R_4 + 2R_2$$

Clearly which is in echelon form.

$\therefore$ The non-zero rows of vectors $\{(1,1,1), (0,1,2)\}$ constitute the L.I set forming the basis of $R(T)$.

$$\Longrightarrow \boxed{\dim R(T) = 2.}$$

Basis for nullspace of T:

$N(T) = \{\alpha \in V_4 \,/\, T(\alpha) = \hat{0}\}$

Let $\alpha \in N(T) \Longrightarrow T(\alpha) = \hat{0}$

$\therefore T(a,b,c,d) = \hat{0}$ where $\hat{0} = (0,0,0) \in V_3$

$\Longrightarrow (a-b+c+d,\ a+2c-d,\ a+b+3c-3d) = (0,0,0)$

$\Longrightarrow a-b+c+d = 0$

$a+2c-d = 0$

$a+b+3c-3d = 0$ we have to solve them for a,b,c,d.

coefficient matrix = $\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 3 & -3 \end{bmatrix}$.

$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 2 & 2 & -4 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$

clearly which is in echelon form.

$\therefore$ The equivalent system of equations are

$a - b + c + d = 0 \Rightarrow \boxed{b = 2d - c}$

$b + c - 2d = 0 \Rightarrow \boxed{a = d - 2c}$

$\therefore$ The number of free variables is 2 namely c, d and the values of a & b depend on ~~these~~ and hence $\boxed{\text{nullity}(T) = \dim(N(T)) = 2}$

$\therefore N(T) = \{(d-2c, 2d-c, c, d) / c, d \in \mathbb{R}\}$

choosing $c = 1, d = 0$, we get $a = -2, b = -1$

$\therefore (a, b, c, d) = (-2, -1, 1, 0)$

choosing $c = 0, d = 1$, we get $a = 1, b = 2$.

$\therefore (a, b, c, d) = (1, 2, 0, 1)$

$\therefore \{(-2, -1, 1, 0), (1, 2, 0, 1)\}$ constitute a basis of $N(T)$.

$\therefore \dim(R(T)) + \dim(N(T)) = 2 + 2$
$= 4 = \dim V_4(\mathbb{R})$

→ verify the Rank-Nullity theorem for the linear map $T: V_4 \to V_3$ defined by $T(e_1) = f_1+f_2+f_3$, $T(e_2) = f_1 - f_2 + f_3$, $T(e_3) = f_1$, $T(e_4) = f_1 + f_3$ where $\{e_1, e_2, e_3\}$ and $\{f_1, f_2, f_3\}$ are standard basis V_4 and V_3 respectively.

Sol: Let $e_1 = (1,0,0,0)$; $e_2 = (0,1,0,0)$; $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$

and $f_1 = (1,0,0)$ $f_2 = (0,1,0)$; $f_3 = (0,0,1)$

$\{e_1, e_2, e_3, e_4\}$ and $\{f_1, f_2, f_3\}$ are the standard basis of V_4 and V_3 respectively.

we have $T(e_1) = f_1 + f_2 + f_3$

$$\Rightarrow T(1,0,0,0) = (1,0,0) + (0,1,0) + (0,0,1)$$
$$= (1,1,1)$$

$$T(e_2) = f_1 - f_2 + f_3$$
$$\Rightarrow T(0,1,0,0) = (1,0,0) - (0,1,0) + (0,0,1)$$
$$= (1,-1,1)$$

$$T(e_3) = f_1$$
$$\Rightarrow T(0,0,1,0) = (1,0,0)$$

$$T(e_4) = f_1 + f_3$$
$$\Rightarrow T(0,0,0,1) = (1,0,0) + (0,0,1)$$
$$= (1,0,1)$$

Let $\alpha \in V_4$

Then α can be written as $\alpha = ae_1 + be_2 + ce_3 + de_4$

Then $T(\alpha) = T(ae_1 + be_2 + ce_3 + de_4)$
$$= aT(e_1) + bT(e_2) + cT(e_3) + dT(e_4)$$
$$= a(1,1,1) + b(1,-1,1) + c(1,0,0) + d(1,0,1)$$
$$= (a+b+c+d,\ a-b,\ a+b+d)$$

consider $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$$\begin{matrix} R_3 \to R_3 - R_1 \\ R_2 \to R_2 - R_1 \end{matrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$R_3 \to R_3 - \frac{1}{2}R_2$, $R_4 \to R_4 - \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ Clearly which is in echelon form.

∴ The non-zero rows of vectors

$\{(1,1,1), (0,-2,0), (0,0,-1)\}$ constitute the L.I set forming the basis of $R(T)$.

$\Rightarrow \boxed{\dim R(T) = 3}$

Basis for null space of T:

$N(T) = \{\alpha \in V_4 \,/\, T(\alpha) = \hat{0}\}$

Let $\alpha \in N(T) \Rightarrow T(\alpha) = \hat{0}$

$\Rightarrow (a+b+c+d, a-b, a+b+d) = (0,0,0)$

$\Rightarrow a+b+c+d = 0$ — (1)

$a-b = 0$ — (2)

$a+b+d = 0$ — (3)

We have to solve for a, b, c, d.

from (1) & (3), we get $\boxed{c=0}$

from (2) & (3) we get $\boxed{d=-2a}$

from (2), we get $\boxed{b=a}$

The number of free variables is 1 namely 'a' and the values d & b depend on 'a' and hence

nullity$(T) = \dim N(T) = 1$.

choosing $a=1$, we get $b=1$, $d=-2$

$(a,b,c,d) = (1,1,0,-2)$.

$\therefore \{(1,1,0,-2)\}$ constitute a basis of $N(T)$.

$\therefore \dim R(T) + \dim N(T) = 3+1$
$= 4$
$= \dim(V_4)$.

HW → Let $T: V_4 \to V_3$ be a linear transformation defined by $T(\alpha_1) = (1,1,1)$; $T(\alpha_2) = (1,-1,1)$;
$T(\alpha_3) = (1,0,0)$; $T(\alpha_4) = (1,0,1)$.
Then verify the $\rho(T) + \nu(T) = \dim V_4$.

→ Find a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^4$ whose range is spanned by $(1,2,0,-4), (2,0,-1,-3)$.

Sol Given ~~the~~ $R(T)$ spanned by
$\{(1,2,0,-4), (2,0,-1,-3)\}$.

Let us include a vector $(0,0,0,0)$ in this set which will not effect the spanning property.

so ~~the~~ $S = \{(1,2,0,-4), (2,0,-1,-3), (0,0,0,0)\}$

Let $B = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of $\mathbb{R}^3$.

W.K.T there exists a transformation T s.t $T(\alpha_1) = (1,2,0,-4)$
$T(\alpha_2) = (2,0,-1,-3)$
$T(\alpha_3) = (0,0,0,0)$

Now if $\alpha \in K \Rightarrow \alpha = (a,b,c) = a\alpha_1 + b\alpha_2 + c\alpha_3$

$\Rightarrow T(\alpha) = T(a,b,c) = T(a\alpha_1 + b\alpha_2 + c\alpha_3)$

$\Rightarrow T(a,b,c) = aT(\alpha_1) + bT(\alpha_2) + cT(\alpha_3)$

$= a(1,2,0,-4) + b(2,0,-1,-3)$
$+ c(0,0,0,0)$

$\therefore T(a,b,c) = (a+2b, 2a, -b, -4a-3b)$

is the reqd transformation.

7) Find $F: \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation whose range is spanned by $(1,-1,2,3)$ and $(2,3,-1,0)$

Sol Consider the standard basis for $\mathbb{R}^3$ is $\{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$.

Then $F(e_1) = (1,-1,2,3)$

$F(e_2) = (2,3,-1,0)$ and

$F(e_3) = (0,0,0,0)$.

W.K.T $(x,y,z) = xe_1 + ye_2 + ze_3$

$F(x,y,z) = F(xe_1 + ye_2 + ze_3)$

$= xF(e_1) + yF(e_2) + zF(e_3)$ ($\because F$ is L.T.)

$= (x, -x, 2x, 3x) + (2y, 3y, -y, 0)$
$+ (0,0,0,0)$

$= (x+2y, -x+3y, 2x-y, 3x)$

Imp) Let V be a vector space of all 2×2 matrices over reals. Let P be a fixed matrix of V; $P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ and $T: V \to V$ be a linear operator defined by $T(A) = PA, A \in V$

find the nullity T.

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$

— The nullspace $N(T)$ is the set of all 2×2 matrices whose T-image is $\bar{0}$.

$\Rightarrow T(A) = PA = \hat{0}$

$\Rightarrow T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-c & b-d \\ a-c & b-d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow a-c=0,\ b-d=0$

$\Rightarrow a=c,\ b=d$

The free variables are c, & d

Hence $\dim N(T) = 2$.

HW. → Describe explicitly the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ whose range space is spanned by $\{(1,0,-1),(1,2,2)\}$.

→ Find the null space, range, rank and nullity of the transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x,y) = (x+y, x-y, y)$

Sol. (i) Let $\alpha = (x,y) \in \mathbb{R}^2$

Then $N(T) = \{\alpha \in \mathbb{R}^2 \mid T(\alpha) = \hat{0}\}$

$\alpha \in N(T) \Rightarrow T(\alpha) = \bar{0}$

$\Rightarrow T(x,y) = \hat{0}$ where $\bar{0} = (0,0,0)$

$\Rightarrow (x+y, x-y, y) = (0,0,0) \in \mathbb{R}^3$.

$$\left.\begin{array}{l} x-y=0 \\ y=0 \end{array}\right\} \Longrightarrow x=0,\ y=0.$$

$\therefore\ \alpha = (x,y) = (0,0) \in \mathbb{R}^2.$

$\therefore$ the nullspace of T consists of only zero vector of $\mathbb{R}^2$.

$\therefore$ nullity $T = \dim N(T) = 0$.

(ii) Range space of $T = \{\beta \in \mathbb{R}^3 / T(\alpha) = \beta$ for $\alpha \in \mathbb{R}^2\}$

$\therefore$ The range space consists of all vectors of the type $(x+y, x-y, y)$ for all $(x,y) \in \mathbb{R}^2$

(iii) $\dim R(T) + \dim N(T) = \dim \mathbb{R}^2$

$\Rightarrow \dim R(T) + 0 = 2$

$\Rightarrow \dim R(T) = 2$

$\Rightarrow$ $\boxed{\text{rank of } T = 2}$

→ show that $T: \mathbb{R}^4 \to \mathbb{R}^4$ given by $T(x,y,z,t) = (2x, 3y, 0, 0)$ is a linear transformation. Find its rank and nullity.

Sol) Let $\alpha = (x_1, y_1, z_1, t_1)$ and $\beta = (x_2, y_2, z_2, t_2)$ be two vectors of $\mathbb{R}^4$.

For $a, b \in \mathbb{R}$

$T(a\alpha + b\beta) = T[a(x_1, y_1, z_1, t_1) + b(x_2, y_2, z_2, t_2)]$

$= T[ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2, at_1 + bt_2]$

$= (2(ax_1 + bx_2), 3(ay_1 + by_2), 0, 0)$ (by the

$= a(2x_1, 3y_1, 0, 0) + b(2x_2, 3y_2, 0, 0)$

$= a\,T(\alpha_1) + b\,T(\beta)$

$\therefore$ T is a linear transformation.

Now we have $N(T) = \{(x, y, z, t) \in \mathbb{R}^4 \,/\, T(x, y, z, t) = (0, 0, 0, 0)\}$

$\therefore (x, y, z, t) \in N(T)$

$\Longleftrightarrow T(x, y, z, t) = (0, 0, 0, 0)$

$\Longleftrightarrow (2x, 3y, 0, 0) = (0, 0, 0, 0)$

$\Longleftrightarrow x = 0, y = 0.$

$\therefore N(T) = \{(0, 0, z, t) \,/\, z, t \in \mathbb{R}\}$

since $(0, 0, z, t) = z(0, 0, 1, 0) + t(0, 0, 0, 1)$

$\therefore N(T)$ is spanned by the set

$S = \{e_3 = (0, 0, 1, 0), e_4 = (0, 0, 0, 1)\}$

Clearly which is L.I.

$\therefore$ S is basis of $N(T)$.

$\therefore \dim N(T) = 2.$

$\boxed{\text{Nullity of } T = 2}$ i.e. $\nu(T) = 2.$

W.K.T

$\dim R(T) + \dim N(T) = \dim \mathbb{R}^4$

$\Rightarrow \dim R(T) + 2 = 4$

$\Rightarrow \boxed{\dim R(T) = 2}.$

HW $\rightarrow$ Show that $T: \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(x, y, z, t) = (x+y, x-y, 0, 0)$ is a linear transformation. Find rank and nullity.

→ Find the range, rank, kernel and nullity of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x_1, x_2, x_3) = (x_1+x_2, 2x_3-x_1)$.

Sol Range space of $T = \{\beta \in \mathbb{R}^2 / T(\alpha) = \beta \text{ for } \alpha \in \mathbb{R}^3\}$

∴ The range space consists of all vectors of the type $(x_1+x_2, 2x_3-x_1)$ for all $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Let $\beta = (a,b) \in R(T)$ be arbitrary,

$T(x_1, x_2, x_3) = (a,b)$ for some $(x_1, x_2, x_3) \in \mathbb{R}^3$.

$\Rightarrow (x_1+x_2, 2x_3-x_1) = (a,b)$

$\Rightarrow (a,b) = (x_1+x_2, 2x_3-x_1)$

$= (x_1+x_2+0x_3, -x_1+0x_2+2x_3)$

$= x_1(1,-1) + x_2(1,0) + 2x_3(0,1)$

Here $S = \{(1,0),(0,1)\}$ is LI and

∴ $R(T) \subseteq L(S)$ & $L(S) \subseteq R(T)$

$\Longrightarrow L(S) = R(T)$.

∴ S is a basis of $R(T)$.

$(1,-1) \in L(S)$.

($\because (1,-1) = (1,0) - (0,1)$)

~~rank~~ $= \dim R(T) = 2$.

Now we have $\ker T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 / T(x_1, x_2, x_3) = (0,0)\}$

Let $(a,b,c) \in \operatorname{Ker} T$ be arbitrary.

Then $T(a,b,c) = (0,0)$

i.e $(a+b, 2c-a) = (0,0)$ (by given)

$\Rightarrow a+b=0 \quad 2c-a=0$

$\Longrightarrow b=-a, \quad c = a/2$

∴ $\operatorname{Ker} T = \{(a, -a, a/2) / a \in \mathbb{R}\}$.

Since $(a, -a, a/2) = a(1, -1, 1/2)$

∴ Ker T is spanned by the set $S = \{(1,-1,\frac{1}{2})\}$

$\therefore S$ is basis of ~~R(~~ $\ker T$.

$\therefore \dim \ker T = 1$

i.e Nullity $T = 1$.

→ Find the range, ~~rank, kernel~~ and nullity of the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$T(x, y, z) = (x+2y-z,\ y+z,\ x+y-2z)$

→ Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x, y, z) = (x-y+2z,\ 2x+y-z,\ -x-2y)$

find the null space of T.

→ Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ defined by

$T(x_1, x_2, x_3) = (2x_1+x_2+x_3,\ x_1+x_2,\ x_1+x_3,\ 3x_1+x_2+2x_3)$.

for each $(x_1, x_2, x_3) \in \mathbb{R}^3$.

Determine a basis for the null space of T. What is the dimension of the Range space of T?

→ Let $T: \mathbb{R}^5 \to \mathbb{R}^5$ be a linear mapping given by

$T(a, b, c, d, e) = (b-d,\ d+e,\ b,\ 2d+e,\ b+e)$.

Obtain bases for its nullspace and range space.

→ Show that $f: \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation where $f(x, y, z) = 3x+y-z$. What is the dimension of the kernel & find a basis for the kernel.

→ Let $T: \mathbb{R}^3 \to \mathbb{R}$ be a linear transformation defined by $T(x, y, z) = (x+y,\ y~~+z~~)$.

find a basis, dimension of each of the range and null space of T.

→ ~~Let~~ $T: V_3(\mathbb{R}) \to V_2(\mathbb{R})$ is defined by

$T(a, b, c) = (a, b)$ $\forall (a, b, c) \in \mathbb{R}^3$.

S.T. T is a linear transformation. find the kernel of T.

Let $V(F)$ be a vectorspace and T be a linear operator on V. Prove that the following statements are true.

(i) The ~~interse~~ction of the range of T and nullspace of T is the zero subspace

i.e $R(T) \cap N(T) = \{\bar{0}\}$.

(ii) If $T[T(\alpha)] = \bar{0}$, then $T(\alpha) = \bar{0}$.

Sol (i) $\Rightarrow$ (ii)

Let $R(T) \cap N(T) = \{\bar{0}\}$.

Let $T(\alpha) = \beta \quad \therefore \beta \in R(T)$ —— (1)

Now $T[T(\alpha)] = \bar{0} \Rightarrow T(\beta) = \bar{0} \Rightarrow \beta \in N(T)$. —— (2)

from (1) & (2) $\beta \in R(T) \cap N(T)$

But $R(T) \cap N(T) = \{\bar{0}\} \Rightarrow \beta = \bar{0}$

$\Rightarrow T(\alpha) = \bar{0}$.

$\therefore T[T(\alpha)] = \bar{0} \Rightarrow T(\alpha) = \bar{0}$.

(ii) $\Rightarrow$ (i) :

Given that $T[T(\alpha)] = \bar{0} \Rightarrow T(\alpha) = \bar{0}$.

Let $\beta \in R(T) \cap N(T)$

$\Rightarrow \beta \in R(T)$ and $\beta \in N(T)$

Now $\beta \in R(T) \Rightarrow T(\alpha) = \beta$ for some $\alpha \in V$

and $\beta \in N(T) \Rightarrow T(\beta) = \bar{0}$

$\Rightarrow T[T(\alpha)] = \bar{0}$

$\Rightarrow T(\alpha) = \bar{0}$

$\Rightarrow \beta = \bar{0} \quad (\because T(\alpha) = \beta)$

$\therefore R(T) \cap N(T) = \{\bar{0}\}$.

Note:- If $T: U \to V$ is a linear transformation then $\rho(T) \leq \min(\dim U, \dim V)$.

→ Is there a linear transformation $T: \mathbb{R}^4 \to \mathbb{R}^2$ for which rank $T = 3$ and nullity $T = 2$?

Sol If $T: \mathbb{R}^4 \to \mathbb{R}^2$ is a linear transformation, then

$$\text{rank}(T) + \text{nullity}(T) = \dim \mathbb{R}^4$$

i.e $3 + 2 = 4$

This is impossible.

Hence T is not a linear transformation.

→ Let T be a linear transformation from $\mathbb{R}^7$ onto a 3-dimensional subspace of $\mathbb{R}^5$. Find $\dim \text{Ker}\, T$.

Sol Let W be a 3-dimensional subspace of $\mathbb{R}^5$ such that $T: \mathbb{R}^7 \to W$ is an onto L.T.

We have

$$T(\mathbb{R}^7) = W \Rightarrow \dim T(\mathbb{R}^7) = \dim W = 3.$$

$$\therefore \text{Rank}(T) = \dim(T(\mathbb{R}^7)) = 3$$

$$\therefore \text{Rank}(T) + \text{Nullity}(T) = \dim \mathbb{R}^7$$

$$\Rightarrow 3 + \text{Nullity}(T) = 7.$$

$$\Rightarrow \text{Nullity}(T) = 7 - 3$$

$$\Rightarrow \text{Nullity}(T) = 4.$$

$$\therefore \dim \text{Ker}\, T = 4.$$

→ Let T be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$ having a 3-dimensional Kernel. find dim Range T.

Sol Given that $T: \mathbb{R}^5 \to \mathbb{R}^3$ is L.T and having a 3-dimensional Kernel.

$\therefore$ dim Ker $T = 3 \Rightarrow$ nullity$(T) = 3$.

$\because$ rank(T) + nullity(T) = dim $\mathbb{R}^5$

$\Rightarrow$ rank$(T) + 3 = 5$

$\Rightarrow$ rank$(T) = 2$

$\Rightarrow \boxed{\text{dim Range } T = 2}$

* Singular and Non-singular Transformation *.

Singular transformation:

A linear transformation $T: U(F) \to V(F)$ is said to be singular if the nullspace of T consists of atleast one non-zero vector.

i.e If there exists a vector $\alpha \in U$ s.t $T(\alpha) = \hat{0}$ for $\alpha \neq \bar{0}$ then T is singular.

* Non-singular Transformation:

A linear transformation $T: U(F) \to V(F)$ is said to be non-singular if the null space consists of one zero vector alone.

i.e $\alpha \in U$ and $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

$\Rightarrow N(T) = \{\bar{0}\}$.

Theorem → Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ be a linear transformation. Then T is non-singular iff the set of images of a linearly independent set is linearly independent.

proof (i) Let T be non-singular and let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a L.I. subset of U. Then its T-images set be $S' = \{T(\alpha_1), T(\alpha_2), \ldots T(\alpha_n)\}$.

Now to prove S' is L.I.

for some $a_1, a_2, \ldots a_n \in F$,

Let $a_1 T(\alpha_1) + a_2 T(\alpha_2) + \cdots + a_n T(\alpha_n) = \hat{0}$ $(\because \hat{0} \in V)$

$\Rightarrow T[a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n] = \hat{0}$ $(\because T \text{ is L.T})$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = \bar{0}$ $(\because T \text{ is non-singular})$

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$ $(\because S \text{ is L.I.})$

$\therefore S'$ is L.I.

(ii) Let the T-images of any L.I. set be L.I. then to prove T is non-singular.

Let $\alpha \in U$ and $\alpha \neq \bar{0}$. Then the set $B = \{\alpha\}$ is L.I. set and image set $B' = \{T(\alpha)\}$ is given to be L.I.

$\Rightarrow T(\alpha) \neq \hat{0}$

$\therefore \alpha \neq \bar{0} \Rightarrow T(\alpha) \neq \hat{0}$

$\therefore T$ is non-singular.

problems → A linear mapping $T: R^3 \to R^3$ is defined by $T(x, y, z) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z)$. Show that T is non-singular.

$\Rightarrow (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta,\ z) = (0,0,0)$

$\Rightarrow x\cos\theta - y\sin\theta = 0$ — (i)

$x\sin\theta + y\cos\theta = 0$ — (ii)

$z = 0$

Squaring and adding eqns (i) & (ii).

$\therefore x^2 + y^2 = 0$

$\Rightarrow x = 0, y = 0$

$\therefore x=0,\ y=0,\ z=0$

$\therefore$ we have $T(x,y,z) = \hat{0}$

$\Rightarrow (x,y,z) = (0,0,0)$

$\therefore$ T is non-singular

→ Show that a linear transformation $T: U \to V$ over the field F is non-singular iff T is one-one

Sol^n (i) Let T be non-singular

i.e, $\alpha \in U$, $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

Now for $\alpha_1, \alpha_2 \in U$,

$T(\alpha_1) = T(\alpha_2)$

$\Rightarrow T(\alpha_1) - T(\alpha_2) = \hat{0}$ $(\because \hat{0} \in V)$

$\Rightarrow T(\alpha_1 - \alpha_2) = \hat{0}$ $(\because T \text{ is L.T})$

$\Rightarrow \alpha_1 - \alpha_2 = \bar{0}$ $(\because T \text{ is non-singular})$

$\Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one-one.

(ii) Let T be one-one.

$\therefore$ zero elt $\hat{0}$ of V is the T-image of only one element $\in U$.

$\Rightarrow$ null space of T consists of only one elt.

Since null space $N(T) \subseteq U$, it must consist of $\bar{0}$.

$\Rightarrow$ null space N(T) consists of only one $\bar{0}$ element

$\Rightarrow N(T) = \{0\}$.

$\Rightarrow T$ is non-singular.

→ Let $T: U \to V$ be a linear transformation of $U(F)$ into $V(F)$ where $U(F)$ is finite dimensional. Prove that U and the range space of T have the same dimension iff T is non-singular.

Sol: (i) Let $\dim U = \dim R(T)$

∴ S.-R.T

$\dim U = \dim R(T) + \dim N(T)$

$\Rightarrow \dim R(T) = \dim R(T) + \dim N(T)$

$\Rightarrow \dim N(T) = 0$

$\Rightarrow$ The null space of T is the zero space $\{\bar{0}\}$.

∴ T is non-singular.

~~(ii) Let~~ T be ~~non-singular~~. Then $N(T) = \{\bar{0}\}$ and nullity $T = 0$ i.e. $\dim(N(T)) = 0$.

As $\dim U = \dim R(T) + \dim N(T)$

$= \dim R(T) + 0$

$\Rightarrow \dim U = \dim R(T)$.

→ If U and V are finite dimensional vector spaces of the same dimension, then a linear mapping $T: U \to V$ is one-one iff it is onto.

Sol: T is one-one $\Leftrightarrow N(T) = \{\bar{0}\}$

$\Leftrightarrow \dim N(T) = 0$

$\Leftrightarrow \dim R(T) + \dim N(T) = \dim U = \dim V$.

$\Leftrightarrow R(T) = V$

$\Leftrightarrow T$ is onto.

Inverse function :

Let $T: U \to V$ be a one-one onto mapping. Then the mapping $T^{-1}: V \to U$ defined by

$T^{-1}(\beta) = \alpha \iff T(\alpha) = \beta$; $\alpha \in U, \beta \in V$; is called the inverse mapping of T.

Note:- If $T: U \to V$ is one-one onto mapping then the mapping $T^{-1}: V \to U$ is also one-one onto.

→ Let $U(F)$ and $V(F)$ be two vector spaces and $T: U \to V$ be a one-one onto linear transformation. Then T^{-1} is a linear transformation and then T is said to be invertible.

Sol. Let $\beta_1, \beta_2 \in V$ and $a, b \in F$

Since T is one-one onto function, $\exists$ unique vectors $\alpha_1, \alpha_2 \in U$ s.t

$$T(\alpha_1) = \beta_1 \text{ and } T(\alpha_2) = \beta_2$$

Hence by the definition of T^{-1}

$$\alpha_1 = T^{-1}(\beta_1) \text{ and } \alpha_2 = T^{-1}(\beta_2)$$

Also $\alpha_1, \alpha_2 \in U$ and $a, b \in F \Rightarrow a\alpha_1 + b\alpha_2 \in U$

$\therefore T(a\alpha_1 + b\alpha_2) = aT(\alpha_1) + bT(\alpha_2)$ ($\because T$ is LT)

$= a\beta_1 + b\beta_2$

$\therefore$ by the defn of inverse

$T^{-1}(a\beta_1 + b\beta_2) = a\alpha_1 + b\alpha_2$

$= aT^{-1}(\beta_1) + bT^{-1}(\beta_2)$

$\therefore T^{-1}$ is a linear transformation from V into U.

→ A linear transformation T on a finite dimensional vectorspace is invertible iff T is non-singular.

Sol: Let U(F) and V(F) be two vectorspaces and have the same dimension.

Let $T: U \to V$ be a linear transformation.

(i) Let T be non-singular.

i.e. for $\alpha \in U$, $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$

Now to prove T is invertible,

it is enough to show T is one-one onto.

Since T is non-singular,

i.e. $\alpha \in U$, $T(\alpha) = \hat{0} \Rightarrow \alpha = \bar{0}$; $N(T) = \{\bar{0}\}$

$\Rightarrow \dim N(T) = 0$.

for $\alpha_1, \alpha_2 \in U$, $T(\alpha_1) = T(\alpha_2)$

$\Rightarrow T(\alpha_1) - T(\alpha_2) = \hat{0}$

$\Rightarrow T(\alpha_1 - \alpha_2) = \hat{0}$ ($\because$ T is LT)

$\Rightarrow \alpha_1 - \alpha_2 = \bar{0}$ ($\because$ T is non-singular)

$\Rightarrow \alpha_1 = \alpha_2$

$\therefore$ T is one-one.

W.K.T $\dim U = \dim R(T) + \dim N(T)$

$= \dim R(T)$ ($\because \dim N(T) = 0$).

Also $T: U \to V$ is one-one

$\Rightarrow V = R(T)$

$\Rightarrow$ T is onto.

(ii) Let T be invertible so that T is one-one onto.

Now to prove T is non-singular.

T is non-singular ⇐ for $\alpha \in U$, $T(\alpha) = \hat{0} = T(\bar{0})$ ($\because$ T is LT)

$\Rightarrow T(\alpha) = T(\bar{0}) \Rightarrow \alpha = \bar{0}$ ($\because$ T is one-one)

NB: → Let $U(F)$ and $V(F)$ be two finite dimensional vector space s.t $\dim U = \dim V$.
If $T: U \to V$ is a linear transformation then the following are equivalent.

1. T is invertible
2. T is non-singular
3. The range of T is V
4. If $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is any basis of U, then $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of V
5. There is some basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of U s.t $\{T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)\}$ is a basis of V.

Here we shall have a series of implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5 \Rightarrow 1$

problems

→ If $T: \mathbb{R}^3 \to \mathbb{R}^3$ is invertible operator defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$.
Find T^{-1}.

Sol. Since T is invertible —

$T(\alpha) = \beta \implies T^{-1}(\beta) = \alpha$; $\alpha \in \mathbb{R}^3$, $\beta \in \mathbb{R}^3$.

Now $T(x, y, z) = (a, b, c) \implies T^{-1}(a, b, c) = (x, y, z)$

Now $(2x, 4x - y, 2x + 3y - z) = (a, b, c)$

$\implies 2x = a,\ 4x - y = b,\ 2x + 3y - z = c$

solving $x = a/2,\ y = 2a - b,\ z = 7a - 3b - c$.

Hence $T^{-1}(a, b, c) = (a/2, 2a - b, 7a - 3b - c)$.

→ The set $\{e_1, e_2, e_3\}$ is the standard basis of $V_3(\mathbb{R})$. $T: V_3(\mathbb{R}) \to V_3(\mathbb{R})$ is a linear operator defined by $T(e_1) = e_1 + e_2$, $T(e_2) = e_2 + e_3$, $T(e_3) = e_1 + e_2 + e_3$.

show that T is non-singular and find its invertible.

Sol Let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

Now $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(e_1) = e_1 + e_2 \Rightarrow T(1,0,0) = (1,1,0)$

$T(e_2) = e_2 + e_3 \Rightarrow T(0,1,0) = (0,1,1)$

$T(e_3) = e_1 + e_2 + e_3 \Rightarrow T(0,0,1) = (1,1,1)$

Let $\alpha = (x,y,z) \in V_3(\mathbb{R})$

$\therefore \alpha = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$\Rightarrow T(\alpha) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$

$= x(1,1,0) + y(0,1,1) + z(1,1,1)$

$\therefore$ The transformation is given by

$T(x,y,z) = (x+z, x+y+z, y+z)$.

Now if $T(x,y,z) = \hat{0}$ then

$(x+z, x+y+z, y+z) = (0,0,0)$

$\Rightarrow x+z=0,\ x+y+z=0,\ y+z=0.$

$\Rightarrow x=y=z=0$

$\therefore T(\alpha)=\hat{0} \Rightarrow \alpha = \hat{0}$

Hence T is non-singular

and therefore T^{-1} exists.

Let $T(x,y,z) = (a,b,c)$ then

$T^{-1}(a,b,c) = (x,y,z)$

Now $(x+z, x+y+z, y+z) = (a,b,c)$

$\Rightarrow x+z=a,\ x+y+z=b,\ y+z=c$

$\Rightarrow \boxed{x = b-c}$ $\boxed{y = b-a}$ $\boxed{z = a-b+c}$

$\therefore T^{-1}(a,b,c) = (x,y,z)$

$= (b-c, b-a, c-b+c)$.

HW → Show that each of the following linear operators T on $\mathbb{R}^3$ is invertible and find T^{-1}.

(a) $T(x,y,z) = (2x, 4x-y, 2x+3y-z)$

(b) $T(a,b,c) = (a-3b-2c, b-4c, c)$

(c) $T(a,b,c) = (3a, a-b, 2a+b+c)$

(d) $T(x,y,z) = (x+y+z, y+z, z)$

(e) $T(a,b,c) = (a-2b-c, b-c, a)$.

HW → The set $\{e_1, e_2, e_3\}$ is the standard basis set of $V_3(\mathbb{R})$. The linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$ is defined below. Show that T is invertible and find T^{-1}.

(i) $T(e_1) = e_1+e_2$, $T(e_2) = e_1-e_2+e_3$, $T(e_3) = 3e_1+4e_3$.

(ii) $T(e_1) = e_1-e_2$, $T(e_2) = e_2$, $T(e_3) = e_1+e_2-7e_3$.

(iii) $T(e_1) = e_1-e_2+e_3$, $T(e_2) = 3e_1-5e_3$, $T(e_3) = 3e_1-2e_3$.

2002 → Show that the mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ where $T(a,b,c) = (a-b, b-c, a+c)$ is linear and non-singular.

Matrix of Linear Transformation

→ Let $U(F)$ and $V(F)$ be two finite dimensional vectorspaces such that $\dim U = n$ and $\dim V = m$. Let $T: U \to V$ be a linear transformation.

Let $B_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the ordered basis of U and $B_2 = \{e_1, e_2, \ldots, e_m\}$ be the ordered basis V.

For every $\alpha \in U \Rightarrow T(\alpha) \in V$ and $T(\alpha)$ can be expressed as a linear combination of elements of of the basis B_2.

If there exists $a's \in F$ s.t

$$T(\alpha_1) = a_{11} e_1 + a_{21} e_2 + \cdots + a_{m1} e_m$$

$$T(\alpha_2) = a_{12} e_1 + a_{22} e_2 + \cdots + a_{m2} e_m$$

$$\cdots$$

$$T(\alpha_j) = a_{1j} e_1 + a_{2j} e_2 + \cdots + a_{mj} e_m \quad \to (A)$$

$$\cdots$$

$$T(\alpha_n) = a_{1n} e_1 + a_{2n} e_2 + \cdots + a_{mn} e_m$$

Writing the co-ordinates $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ successively as columns of a matrix we get,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ - & - & \cdots & \cdots & \cdots & - \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

This matrix represented as $[a_{ij}]_{m\times n}$ is called the matrix of the linear transformation T w.r.t to the bases B_1 and B_2.

Symbolically $[T : B_1; B_2]$ or $[T] = [a_{ij}]_{m\times n}$.

Thus the matrix $[a_{ij}]_{m\times n}$ completely determines the linear transformation through the relations given in (A).

Hence the matrix $[a_{ij}]_{m\times n}$ represents the transformation T.

Imp **Note:-** Let $T: V \to V$ be a linear operator so that $\dim V = n$.

If $B_1 = B_2 = B$ (say) then the above said matrix is called the matrix of T relative to the ordered basis B.

It is denoted by $[T ; B] = [T]_B = [a_{ij}]_{n\times n}$.

problems

1) Let $T: V_2 \to V_3$ be defined by

$T(x,y) = (x+y, 2x-y, 7y)$

find $[T : B_1, B_2]$ where B_1 and B_2 are the standard bases of V_2 and V_3.

sol. B_1 is standard basis of V_2 and B_2 is standard basis of V_3.

$\therefore B_1 = \{(1,0), (0,1)\}$

$B_2 = \{(1,0,0), (0,1,0), (0,0,1)\}$.

Now $T(1,0) = (1,2,0)$

$= 1(1,0,0) + 2(0,1,0) + 0(0,0,1)$

$T(0,1) = (1,-1,7)$

$= 1(1,0,0) - 1(0,1,0) + 7(0,0,1)$.

bases B_1 and B_2 is

$$[T; B_1, B_2] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 0 & 7 \end{bmatrix}.$$

→ Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by

$T(x, y, z) = (3x+2y-4z, x-5y+3z)$.

Find the matrix of T relative to the bases $B_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$

$B_2 = \{(1,3), (2,5)\}$.

Sol Let $(a,b) \in \mathbb{R}^2$ and

Let $(a,b) = p(1,3) + q(2,5)$

$= (p+2q, 3p+5q)$

$\Rightarrow p+2q = a, \; 3p+5q = b$

Solving $p = -5a+2b, \; q = 3a-b$

$\therefore (a,b) = (-5a+2b)(1,3) + (3a-b)(2,5)$ — (1)

Now $T(1,1,1) = (1,-1)$

$= -7(1,3) + 4(2,5)$ (from (1))

$T(1,1,0) = (5,-4)$

$= -33(1,3) + 19(2,5)$ (from (1))

$T(1,0,0) = (3,1)$

$= -13(1,3) + 8(2,5)$ (from (1))

$\therefore$ The matrix of L.T relative to B_1 and B_2 is

$[T: B_1, B_2] = \begin{bmatrix} -7 & -33 & -13 \\ 4 & 19 & 8 \end{bmatrix}$

→ If the matrix of a linear operator T on $V_3(\mathbb{R})$ w.r.t the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$ Describe explicitly $T: V_3(\mathbb{R}) \to V_3$

What is the matrix of T w.r.t the basis $\{(0,1,-1), (1,-1,1), (-1,1,0)\}$.

Sol (i) Let the standard basis of $V_3(\mathbb{R})$ be

$B = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $\alpha_1 = (1,0,0), \alpha_2 = (0,1,0), \alpha_3 = (0,0,1)$

$\therefore$ Given $[T]_B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$

$\therefore T(\alpha_1) = 0\,\alpha_1 + 1\,\alpha_2 + (-1)\alpha_3$

$= 0(1,0,0) + 1(0,1,0) + (-1)(0,0,1)$

$= (0, 1, -1)$

$T(\alpha_2) = 1\cdot\alpha_1 + 0\cdot\alpha_2 + (-1)\alpha_3 = (1, 0, -1)$

$T(\alpha_3) = 1\cdot\alpha_1 + (-1)\alpha_2 + 0\cdot\alpha_3 = (1, -1, 0)$

Let $(a,b,c) \in V_3(\mathbb{R})$ then

$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$= a\alpha_1 + b\alpha_2 + c\alpha_3$.

$\therefore T(a,b,c) = aT(\alpha_1) + bT(\alpha_2) + cT(\alpha_3)$

$= a(0,1,-1) + b(1,0,-1) + c(1,-1,0)$

$= (b+c, a-c, -a-b)$

which is the reqd L.T.

(ii) Let $B_2 = \{\rho_1, \rho_2, \rho_3\}$ where

$\rho_1 = (0,1,-1)$, $\rho_2 = (1,-1,1)$, $\rho_3 = (-1,1,0)$

Using the transformation

$\therefore T(a,b,c) = (b+c, a-c, -a-b)$,

we have

$$T(\rho_1) = T(0,1,-1) = (0,1,-1)$$

$$T(\rho_2) = T(1,-1,1) = (0,0,0)$$

$$T(\rho_3) = T(-1,1,0) = (1,-1,0)$$

Now let $(a,b,c) = x\rho_1 + y\rho_2 + z\rho_3$

$$= x(0,1,-1) + y(1,-1,1) + z(-1,1,0)$$

$$= (y-z, x-y+z, -x+y)$$

$$\Rightarrow \begin{matrix} y-z=a \\ x-y+z=b \\ -x+y=c \end{matrix} \Rightarrow \begin{matrix} \boxed{x = a+b} \\ \boxed{y = a+b+c} \\ \boxed{z = b+c} \end{matrix}$$

$\therefore (a,b,c) = (a+b)\rho_1 + (a+b+c)\rho_2 + (b+c)\rho_3$ — (1)

$\therefore T(\rho_1) = (0,1,-1)$

$= 1\cdot\rho_1 + 0\cdot\rho_2 + 0\cdot\rho_3$ (from (1))

$T(\rho_2) = (0,0,0) = 0\cdot\rho_1 + 0\cdot\rho_2 + 0\cdot\rho_3$

$T(\rho_3) = (1,-1,0) = 0\cdot\rho_1 + 0\cdot\rho_2 - 1\cdot\rho_3$.

$$\therefore [T; B_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

→ Let $D: P_3 \to P_2$ be the polynomial differential transformation $D(p) = \frac{dp}{dx}$. Find the matrix of D relative to the standard bases.

$B_1 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$.

Sol $D(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$

$D(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$D(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$

$D(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2$.

∴ The matrix of D relative to B_1 and B_2 is $[T: B_1, B_2]$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

2005 → Let T be a linear transformation on $\mathbb{R}^3$, whose matrix relative to the standard basis of $\mathbb{R}^3$ is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}.$$

Find the matrix of T relative to the basis $B = \{(1,1,1), (1,1,0), (0,1,1)\}$.

2007 → If $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$T(x, y) = (2x - 3y, x + y)$.

Compute the matrix of T relative to the basis $B = \{(1,2), (2,3)\}$.

1. Let $R_3[x] = \{a_0 + a_1 x + a_2 x^2 / a_0, a_1, a_2 \in \mathbb{R}\}$.

Define $T : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ by $T(f(x)) = \frac{d}{dx} f(x)$ — (1)

for all $f(x) \in \mathbb{R}_3[x]$. Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1+x, 1+x+x^2\}$.

Sol Let $f(x), g(x) \in \mathbb{R}_3[x]$ and $a, b \in \mathbb{R}$

By (1), we have

$$T(a f(x) + b g(x)) = \frac{d}{dx}(a f(x) + b g(x))$$

$$= a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x)$$

$$= a T(f(x)) + b T(g(x))$$

$\therefore$ T is a linear transformation.

Now $T(1) = \frac{d}{dx}(1) = 0$

$T(x) = \frac{d}{dx}(x) = 1$

$T(x^2) = \frac{d}{dx}(x^2) = 2x$

Again $T(1) = 0 = 0\cdot 1 + 0x + 0x^2$

$T(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$

$T(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$

Hence the matrix representation of T w.r.t. the basis $\{1, x, x^2\}$ is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Now $T(1) = \frac{d}{dx}(1) = 0$

$T(1+x) = \frac{d}{dx}(1+x) = 1$

$T(1+x+x^2) = \frac{d}{dx}(1+x+x^2) = 1+2x.$

Again $T(1) = 0 = 0\cdot 1 + 0(1+x) + 0(1+x+x^2)$

$T(1+x) = 1 = 1\cdot 1 + 0\cdot(1+x) + 0(1+x+x^2)$

$T(1+x+x^2) = 1+2x = (-1)\cdot 1 + 2(1+x) + 0(1+x+x^2)$

Hence the matrix representation of T w.r.t the basis $\{1, 1+x, 1+x+x^2\}$ is

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

H.W → Let $R_4[x] = \{a_0 + a_1x + a_2x^2 + a_3x^3 / a_i \in \mathbb{R}\}$.

Define $T: R_4[x] \to R_4[x]$ as

$T(f(x)) = \frac{d}{dx}(f(x))$ for all $f(x) \in R_4[x]$.

Let $\beta = \{1, x, x^2, x^3\}$ be an ordered basis $R_4[x]$. Find $[T]_\beta$.

→ Let V be the vector space polynomials of degree ≤ 3 over F. Let T be a linear transformation defined on V by $T(a_0 + a_1x + a_2x^2 + a_3x^3) =$

$a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$. (1)

Compute the matrix of T relative to the bases (a) $\{1, x, x^2, x^3\}$ (b) $\{1, 1+x, 1+x^2, 1+x^3\}$.

Sol

(a) $T(1) = 1 = 1 + 0x + 0x^2 + 0x^3$

$T(x) = x+1 = 1 + 1\cdot x + 0x^2 + 0\cdot x^3$

$T(x^2) = (x+1)^2 = 1 + 2x + 1\cdot x^2 + 0\cdot x^3$

$T(x^3) = (x+1)^3 = 1 + 3x + 3x^2 + 1\cdot x^3.$

Hence the matrix representation of T w.r.t the basis $\{1, x, x^2, x^3\}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) $T(1) = 1$

$T(1+x) = 1 + (1+x)$

$T(1+x^2) = 1 + (x+1)^2 = 1 + (1 + x^2 + 2x)$

$T(1+x^3) = 1 + (x+1)^3$
$= 1 + (x^3 + 1 + 3x^2 + 3x)$

Again $T(1) = 1 = 1\cdot 1 + 0\cdot(1+x) + 0\cdot(1+x^2) + 0\cdot(1+x^3).$

$T(1) = 1 = 1\cdot 1 + 1\cdot(1+x) + 0(1+x^2) + 0(1+x^3).$

$T(1+x) = 1 + (1+x) = 1\cdot 1 + 1\cdot(1+x) + 0(1+x^2) + 0\cdot(1+x^3)$

$T(1+x^2) = -1\cdot 1 + 2(1+x) + 1\cdot(1+x^2) + 0\cdot(1+x^3)$

$T(1+x^3) = 1 + (1+x^3) + (3x + 3x^2)$
$= -5(1) + 3(1+x) + 3(1+x^2) + 1\cdot(1+x^3)$

Hence the matrix representation of T w.r.t the basis $\{1, 1+x, 1+x^2, 1+x^3\}$ is

(P.T.O)

$$\begin{bmatrix} 1 & 1 & -1 & -5 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Consider the vector space

$X := \{ p(x) \mid p(x)$ is a polynomial of degree less than or equal to 3 with real coefficients$\}$ over the real field $\mathbb{R}$. Define the map $D : X \to X$

by $D(p(x)) := p_1 + 2p_2 x + 3p_3 x^2$;

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$.

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the ordered basis $\{1, x, x^2, x^3\}$ for X.

Sol Let $p(x), q(x)$, $a, b \in \mathbb{R}$.

Given map D defined by

$$D(p(x)) = p_1 + 2p_2 x + 3p_3 x^2$$

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

i.e $D(p_0 + p_1 x + p_2 x^2 + p_3 x^3) = p_1 + 2p_2 x + 3p_3 x^2$. — (1)

$D = \frac{d}{dx}$

Now $D[a p(x) + b q(x)] = D[a(p_0 + p_1 x + p_2 x^2 + p_3 x^3) + b(q_0 + q_1 x + q_2 x^2 + q_3 x^3)]$

$= D[(ap_0 + bq_0) + (ap_1 + bq_1)x + (ap_2 + bq_2)x^2 + (ap_3 + bq_3)x^3]$

$= (ap_1 + bq_1) + 2(ap_2 + bq_2)x + 3(ap_3 + bq_3)x^2$

$= a(p_1 + 2p_2 x + 3p_3 x^2) + b(q_1 + 2q_2 x + 3q_3 x^2)$

$= a\,D(p(x)) + b\,D(q(x))$

$\therefore D: X \to X$ is a linear transformation.

Now

From ①,

$D(1) = 0 = 0 + 0x + 0x^2 + 0x^3$

$D(x) = 1 = 1 + 0x + 0x^2 + 0x^3$

$D(x^2) = 2x = 0 + 2\cdot x + 0x^2 + 0x^3$

$D(x^3) = 3x^2 = 0 + 0x + 0x^2 + 3\cdot x^2$

Hence the matrix representation of D w.r.t the ordered basis $\{1, x, x^2, x^3\}$

is $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$